1. (Chapter 2)

(a) (5%) Is the system with input \( x(t) \), output \( y(t) \) and input-output relation

\[
y(t) = \text{Re} \left\{ x(t)e^{2\pi f_0 t} \right\}
\]

a linear system? If the answer is positive, prove it. If negative, give a counterexample.

Hint: Superposition principle.

(b) (5%) Is the system in (a) time-invariant? If the answer is positive, prove it. If negative, give a counterexample.

Hint: For a time-invariant system, input \( x(t - \tau) \) should induce output \( y(t - \tau) \) if input \( x(t) \) induces output \( y(t) \).

(c) (5%) Give a counterexample (i.e., give an example of \( f_0 \) and \( X_\ell(f) \)) that fails the linear-time-invariant (LTI) system below:

\[
\text{Input } x(t) = \text{Re} \left\{ x_\ell(t)e^{2\pi f_0 t} \right\},
\]

\[
\text{Output } y(t) = \text{Im} \left\{ x_\ell(t)e^{2\pi f_0 t} \right\},
\]

Transfer function \( H(f) = -i \cdot \text{sgn}(f) \)

Hint: Consider the relation between \( f_0 \) and the bandwidth \( W \) of \( x_\ell(t) \).

(d) (5%) Prove that the power spectrum density of a WSS process \( x(t) \) is always real-valued.

Hint: The autocorrelation function \( R_s(\tau) \) satisfies \( R_s(-\tau) = R_s^*(\tau) \).

(e) (5%) Show that \( x(t) = \text{Re} \left\{ x_\ell(t)e^{2\pi f_0 t} \right\} \) implies

\[
X(f) = \frac{1}{2} [X_\ell(f - f_0) + X_\ell^*(-f - f_0)].
\]

2. (Chapter 3)

(a) (5%) Suppose \( s(t) \) is a cyclostationary random process with period \( T \). Let random vector \( \bar{s}(t) \) be defined as:

\[
\bar{s}(t) = \begin{bmatrix} s(t) \\ s(t - \tau) \end{bmatrix},
\]

where \( \tau \) is a constant. Is \( \bar{s}(t) \) also cyclostationary? If your answer is positive, prove it. If negative, show a counterexample.

(b) (5%) If the autocorrelation function of \( s(t) \) is

\[
R_s(t_1, t_2) = \sum_{m=-\infty}^{\infty} g(t_1 - mT)g^*(t_2 - mT),
\]

where \( g(t) \) is a given continuous waveform. Prove that the time-average autocorrelation function of \( s(t) \) is given by:

\[
\bar{R}_s(\tau) = \frac{1}{T} \int_{-\infty}^{\infty} g(u + \tau)g^*(u)du.
\]
(c) (5%) Following (b), further prove that the time-average power spectrum density of \( s(t) \) is equal to:

\[
\overline{S}_s(f) = \frac{1}{T} |G(f)|^2.
\]

(d) (6%) Below is the passband signal of OQPSK modulation.

\[
s_{\text{OQPSK}}(t) = \sum_{n=-\infty}^{\infty} I_{2n} g(t - 2nT) \cos(2\pi f_c t) - \sum_{n=-\infty}^{\infty} I_{2n+1} g(t - (2n + 1)T) \sin(2\pi f_c t)
\]

where \( g(t) = \begin{cases} 1, & 0 \leq t < 2T \\ 0, & \text{otherwise} \end{cases} \) and \( I_n \in \{\pm 1\} \). Assume that \( T \) is a multiple of \( 1/f_c \). Define the inner product of two signals, \( a(t) \) and \( b(t) \), to be \( \int_0^T a(t)b(t)dt \). Now by considering all possible signal waveforms that could appear during \([0, T)\), determine the dimension of the OQPSK modulation (3%). Also, give an orthonormal basis of these waveforms (3%).

Hint: \( s_{\text{OQPSK}}(t) = I_0 g(t) \cos(2\pi f_c t) - I_{-1} g(t + T) \sin(2\pi f_c t) \).

(e) (6%) Re-do subproblem (d) by re-defining \( g(t) \) to be \( g(t) = \begin{cases} \sin \left( \pi \frac{t}{2T} \right), & 0 \leq t < 2T \\ 0, & \text{otherwise} \end{cases} \).

Hint: You may let \( f_1 = f_c - \frac{1}{4T} \) and \( f_2 = f_c + \frac{1}{4T} \) for notational convenience.

3. (Chapter 4) Consider two signals defined as

\[
s_1(t) = \begin{cases} A, & 0 \leq t < T \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad s_2(t) = \begin{cases} A, & 0 \leq t < \tau \\ -A, & \tau \leq t < T \\ 0, & \text{otherwise} \end{cases}
\]

(a) (5%) Find signal space representations of \( s_1(t) \) and \( s_2(t) \) based on the basis

\[
\phi_1(t) = \begin{cases} \frac{1}{\sqrt{\tau}}, & 0 \leq t < \tau \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \phi_2(t) = \begin{cases} \frac{1}{\sqrt{T - \tau}}, & \tau \leq t < T \\ 0, & \text{otherwise} \end{cases}
\]

Note that the inner product between two signals \( a(t) \) and \( b(t) \) is defined as

\[
\int_0^T a(t)b(t)dt.
\]

(b) (5%) Use the two signals to carry binary information over the AWGN channel with one-sided power spectrum density \( N_0 \) (in other words, the power spectrum density of the additive white noise is equal to \( N_0/2 \)). What are the optimal decision regions for the two signals. Suppose \( s_1(t) \) will be used with probability \( p \).

(c) (5%) Find the optimal error probability in (b).

Hint: \( \Pr \{N(m, \sigma^2) < r\} = Q \left( \frac{m - r}{\sigma} \right) \).

(d) (6%) Now let \( \tau = T/2 \) for \( \phi_1(t) \) and \( \phi_2(t) \) in (a). Consider the following constellation. Assume equal prior probability. Using the fact that the binary error probability between points \( s_i \) and \( s_j \) can be expressed as \( Q \left( d_{i,j}/\sqrt{2N_0} \right) \), find the union bound expression in terms of every \( d_{i,j} \) and \( Q \)-function for the error probability of this constellation, where \( d_{i,j} \) is the Euclidean distance between points \( s_i \) and \( s_j \).
Hint: The distance enumerator function of this constellation is given by
\[ T(X) = 16X^d + 8X^{2d} + 12X^{4d} + 16X^{5d} + 4X^{8d}. \]

(e) (6%) Following (d), find the union bound expression in terms of \( d_{\min} = \min_{i \neq j} d_{i,j} \) and Q-function for the error probability of this constellation.

4. (Chapter 4)

(a) (11%) Consider four equal-probable signals,
\[
\begin{align*}
    s_1 &= \begin{bmatrix} -1 \\ -1 \end{bmatrix}, &
    s_2 &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}, &
    s_3 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, &
    s_4 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\end{align*}
\]
sending through an additive noisy channel with \( n = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \) with joint pdf
\[
    f(n) = \begin{cases} 
        \exp(-n_1 - n_2), & \text{if } n_1, n_2 \geq 0 \\
        0, & \text{otherwise}. 
    \end{cases}
\]
Find MAP rule for the four symbols (8%), and symbol error rate \( P_e \) (3%).

Hint: For your convenience,
\[
\int_{-1}^{1} \int_{-1}^{1} e^{-r_1-r_2} = (e - e^{-1})^2, \quad \int_{1}^{\infty} \int_{-1}^{1} e^{-r_1-r_2} = 1 - e^{-2}, \quad \text{and} \quad \int_{1}^{\infty} \int_{1}^{\infty} e^{-r_1-r_2} = e^{-2}.
\]
Note that
\[
\begin{align*}
    f(r|s_1) &= e^{-r_1-r_2-2} \cdot 1\{r_1 \geq -1, r_2 \geq -1\}, \quad f(r|s_2) = e^{-r_1-r_2} \cdot 1\{r_1 \geq -1, r_2 \geq 1\} \\
    f(r|s_3) &= e^{-r_1-r_2} \cdot 1\{r_1 \geq 1, r_2 \geq -1\}, \quad f(r|s_4) = e^{-r_1-r_2+2} \cdot 1\{r_1 \geq 1, r_2 \geq 1\}.
\end{align*}
\]
(b) (6%) Determine the corresponding bit error rate (respectively for the two bits) if the receiver does the following bit mapping after the symbol detection in (a).

\[ s_1 \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad s_2 \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad s_3 \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad s_4 \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

Note: For notational convenience, you may denote the bit error rates respectively for the first and second bits as \( P_{b,1} \) and \( P_{b,2} \).

(c) (4%) Re-do subproblem (b) for the new bit mapping below.

\[ s_1 \mapsto \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad s_2 \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad s_3 \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad s_4 \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \]

(d) (5% bonus) Explain why one of the bit mappings in (b) and (c) performs better than the other one.