The above figure shows an equivalent channel model with channel transfer function $C_\ell(f)$ bandlimited to $W$ and random, and $z_W(t)$ bandlimited white noise. It can be formulated as:

$$r_\ell(t) = \int_{-\infty}^{\infty} s_\ell(f)C_\ell(f)e^{j2\pi ft}df + z_W(t),$$

where $s_\ell(f)$ is the Fourier transform of $s_\ell(t)$.

(a) (6%) Since $C_\ell(f)$ is bandlimited to bandwidth $W$, its impulse response can be expressed using the samples $\{c_n = \frac{1}{W}c_\ell\left(\frac{n}{W}\right)\}_{n=-\infty}^{\infty}$, i.e.,

$$c_\ell(t) = W \sum_{n=-\infty}^{\infty} c_n \cdot \text{sinc}\left(W(t - \frac{n}{W})\right).$$

Show that $C_\ell(f)$ can also be expressed using the samples $\{c_n\}_{n=-\infty}^{\infty}$.

Hint: $\int_{-\infty}^{\infty} W \cdot \text{sinc}(Wt)e^{-j2\pi ft}dt = \begin{cases} 1, & |f| < \frac{W}{2} \\ 0, & \text{otherwise} \end{cases}$

(b) (6%) Suppose $c_n = 0$ with probability one for $n \leq 0$ and $n \geq 4$ (i.e., only $c_1$, $c_2$ and $c_3$ are possibly non-zero). We can then derive:

$$r_\ell(t) = \int_{-\infty}^{\infty} s_\ell(f)C_\ell(f)e^{j2\pi ft}df + z_W(t)$$

$$= \sum_{n=1}^{3} c_n \int_{-W/2}^{W/2} s_\ell(f)e^{j2\pi f(t-n/W)}df + z_W(t)$$

$$= \sum_{n=1}^{3} c_n \cdot s_\ell\left(t - \frac{n}{W}\right) + z_W(t).$$

Draw the equivalent tapped-delay-line channel model diagram of the figure above.

(c) (6%) By borrowing the idea from (b), draw the equivalent tapped-delay-line channel model diagram of a general three-path fading channel illustrated below, where we assume $\tau_2 < \tau_1 < \tau_3$. 

1.
Hint: You may wish to replace the delay component \( \frac{1}{W} \) in subproblem (b) with the general delay component \( \tau \) for some proper delay \( \tau \).

2. (a) (6%) Give the PSDs of a broadband interference and a CW jammer.
   (b) (6%) Give the definition of Jamming Margin.
   (c) (6%) Explain what the pulsed interference is.

3. A 2-user CDMA system can be modeled as:

\[
\begin{align*}
    r_i &= (2c_{m,1,i} - 1)(2b_{1,i} - 1)2E_c + (2c_{m,2,i} - 1)(2b_{2,i} - 1)2E_c + \nu_i, \quad i = 1, 2, \ldots, n \\
    &= \nu_i'
\end{align*}
\]

where each \( c_{m,1,i}, c_{m,2,i}, b_{1,i}, b_{2,i} \in \{0, 1\}, 1 \leq m \leq M = 2, \) and \( \{\nu_i\}_{i=1}^n \) zero-mean Gaussian i.i.d. with \( \mathbb{E}[\nu_i^2] = 2J_0E_c \). In order to recover the information \( m_1 \) from the first user, the receiver will perform

\[
y_i = (2b_{1,i} - 1) r_i = (2c_{m,1,i} - 1)2E_c + (2c_{m,2,i} - 1)(2b_{2,i} - 1)(2b_{1,i} - 1)2E_c + (2b_{1,i} - 1) \nu_i. \quad (2)\]

(a) (6%) Give one example of pseudo-random sequences \( (b_{1,i}, \ldots, b_{1,n}) \) and \( (b_{2,i}, \ldots, b_{2,n}) \) with \( n \geq 7 \) such that

\[
\sum_{i=1}^n (2b_{1,i} - 1)(2b_{2,i} - 1) = -1. \quad (3)
\]

Hint: Maximum-length shift register sequence or m-sequence with \( m \geq 3 \).

(b) (8%) Suppose

\[
(c_{m,1}, \ldots, c_{m,n}) = \begin{cases} 
    000 \cdots 000, & m = 1 \\
    111 \cdots 111, & m = 2
\end{cases}
\]

Under equal prior probability and the simplified treatment that \( \{\nu_i'\}_{i=1}^n \) is zero-mean i.i.d. Gaussian distributed, i.e., (2) is simplified to

\[
y_i = (2c_{m,1,i} - 1)2E_c + \nu_i',
\]

the optimal decision rule is given by:

\[
\hat{m} = \arg \min_{1 \leq m \leq 2} \|y - 2E_c(2c_m - 1)\|^2 = \arg \max_{1 \leq m \leq 2} \sum_{i=1}^n (2c_{m,i} - 1)y_i. \quad (4)
\]
Prove that the error rate under decision rule (4) is equal to

\[ P_{e,\text{sim}} = Q \left( \sqrt{\frac{2E_c}{2E_c + J_0}} \right), \]

provided that the i.i.d. Gaussian \( \{\nu'_i\}_{i=1}^n \) has mean

\[ \mathbb{E} [(2c_{m,i} - 1)(2b_{2,i} - 1)(2b_{1,i} - 1)2E_c + (2b_{1,i} - 1)\nu_i] \]

and variance

\[ \text{Var} [(2c_{m,i} - 1)(2b_{2,i} - 1)(2b_{1,i} - 1)2E_c + (2b_{1,i} - 1)\nu_i], \]

and \( \{c_{m,i}\}_{i=1}^n, \{b_{1,i}\}_{i=1}^n \) and \( \{b_{2,i}\}_{i=1}^n \) are uniform i.i.d. sequences and are independent to each other.

**Hint:** Under

\[ \sum_{i=1}^{n} (2c_{m,i} - 1) y_i = (2m - 3) \sum_{i=1}^{n} y_i = (2m - 3) \left( n(2m_1 - 3)2E_c + \sum_{i=1}^{n} \nu'_i \right), \]

derive \( \text{Pr}[\hat{m} = 1|m_1 = 2] \) and \( \text{Pr}[\hat{m} = 2|m_1 = 1] \). Note that \( \text{Pr}\{N(m, \sigma^2) < r\} = Q \left( \frac{m-r}{\sigma} \right). \)

(c) (8%) The simplified treatment in subproblem (b) is actually unreal since both \( \{b_{1,i}\}_{i=1}^n \) and \( \{b_{2,i}\}_{i=1}^n \) are known constant sequences to the receiver, and \( \{c_{m,i}\}_{i=1}^n \) is constant although unknown to the receiver. Prove that the error rate under the same decision rule in subproblem (b) is actually equal to

\[ P_e = \frac{1}{2} Q \left( \sqrt{2\alpha^2 \frac{2E_b}{J_0}} \right) + \frac{1}{2} Q \left( \sqrt{2(2 - \alpha)^2 \frac{2E_b}{J_0}} \right), \]

where \( E_b = nE_c \), provided that the two pseudo-random sequences satisfy

\[ \sum_{i=1}^{n} (2b_{1,i} - 1)(2b_{2,i} - 1) = (1 - \alpha)n \text{ for some } 0 \leq \alpha \leq 2. \]  

(5)

**Hint:** Under

\[ \sum_{i=1}^{n} (2c_{m,i} - 1) y_i = (2m - 3) \sum_{i=1}^{n} y_i = (2m - 3) \left( n(2m_1 - 3)2E_c + \sum_{i=1}^{n} (2b_{1,i} - 1)\nu_i \right), \]

derive \( \text{Pr}[\hat{m} = 1|m_1 = 2] \) and \( \text{Pr}[\hat{m} = 2|m_1 = 1] \).

4. (a) (8%) Suppose that random processes \( p_{PN}(t) \) and \( z(t) \) are independent to each other, and \( z(t) \) is a wide-sense stationary process. Show that the time-averaged autocorrelation function of the product random signal \( p_{PN}(t)z(t) \) is given by

\[ \bar{R}_{p\times z}(\tau) = R_z(\tau)\bar{R}_p(\tau) \]

where \( R_z(\tau) \) is the autocorrelation function of \( z(t) \), and \( \bar{R}_p(\tau) \) is the time-averaged autocorrelation function of \( p_{PN}(t) \).

**Hint:** No statistical assumption on \( p_{PN}(t) \) is given except that its time-averaged PSD exists.
(b) (8%) From Chapter 3, we learn that the time-averaged PSD of
\[ c(t) = \sum_{n=-\infty}^{\infty} I_n s(t - nT_b) \]
is equal to
\[ \bar{S}_c(f) = \frac{1}{T_b} S_I(f) |S(f)|^2 \]
where \( S_I(f) = \sum_{k=-\infty}^{\infty} R_I(k)e^{-i2\pi kfT_c} \) is the PSD of \( \{I_n\}_{n=-\infty}^{\infty} \), \( R_I(k) = \mathbb{E}[I_{m+k}I_m^*] \) is the autocorrelation function of WSS \( \{I_n\}_{n=-\infty}^{\infty} \), and \( S(f) = \int_{-\infty}^{\infty} s(t)e^{-i2\pi ft}dt \) is the spectrum of \( s(t) \). Prove that
\[ \bar{S}_c(f) = \frac{2}{T_b} |G(f)|^2(1 + \cos(2\pi fT_c)) \]
if \( \{I_n\}_{n=-\infty}^{\infty} \) are uniform i.i.d. with \( I_n \in \{-1, 1\} \), and
\[ s(t) = \begin{cases} g(t \mod T_c) & 0 \leq t < T_b \\ 0 & \text{otherwise} \end{cases} \]
where \( g(t) \neq 0 \) only when \( 0 \leq t < T_c \), and \( T_b/T_c = 2 \).

**Hint:** Note that \( s(t) = \sum_{n=0}^{1} g(t - nT_c) \), and \( G(f) = \int_{0}^{T_c} g(t)e^{-i2\pi ft}dt \).

(c) (8%) Continue from (b). After spreading \( c(t) \) with (deterministically chosen) PN sequence \( (b_0, b_1) = (1, 0) \) (which accordingly satisfies the balanced property), we have
\[ p_{PN}(t)c(t) = \sum_{i=-\infty}^{\infty} (2b_i - 1)I_{\lfloor i/2\rfloor}p(t - iT_c)s(t - 2\lfloor i/2\rfloor T_c), \]
where \( p(t) \) is a rectangular pulse of height 1 and duration \( T_c \), and \( b_i = b_i \mod 2 \). Prove that
\[ \bar{S}_{p\times c}(f) = \frac{2}{T_b} |G(f)|^2(1 - \cos(2\pi fT_c)) \]

**Hint:** In this derivation, we treat the PN sequence as a deterministic sequence rather than the textbook-assumed statistical sequence. Find \( \tilde{g}(t) \) such that
\[ p_{PN}(t)c(t) = \sum_{k=-\infty}^{\infty} I_k \tilde{g}(t - kT_b) \]
Then
\[ \bar{S}_{p\times c}(f) = \frac{1}{T_b} S_I(f)|\tilde{G}(f)|^2. \]

5. For a lowpass signal of the form,
\[ s_\ell(t) = \kappa \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{Q-1} X_{k,n}e^{i2\pi k t/T} \right) g(t - nT) \]
answer the following questions.
(a) (6%) Explain that why the signal can be regarded as a $Q$-carrier system with single-carrier implementation. What are the $Q$ carrier frequencies in this system?

(b) (8%) If \( \{X_{k,n}\}_{-\infty<k,n<\infty} \) are zero-mean i.i.d. with variance \( \sigma^2 \), show that

\[
S_{s_{\ell}}(f) = \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} \left| G \left( f - \frac{k}{T} \right) \right|^2.
\]

(c) (6%) Why adding cyclic prefix (CP) to the signal \( s_{\ell}(t) \) before its transmission?

*Hint: It is relevant to the channel impulse response \( c_{\ell}(t) \).*

(d) (8%) Show that with CP technique, the noiseless reception from the channel with impulse response \( c_{\ell}(t) \) is given by

\[
r_{\ell}(t) = \kappa \sum_{k=0}^{Q-1} C_{\ell} \left( \frac{k}{T} \right) X_k e^{j2\pi km/N},
\]

where \( C_{\ell}(f) \) is the spectrum of \( c_{\ell}(t) \).

*Hint: \( r_{\ell}(t) = \tilde{s}_{\ell}(t) \ast c_{\ell}(t) \), where \( \tilde{s}_{\ell}(t) \) is the periodic counterpart of \( s_{\ell}(t) \).*