5.1 Signal parameters estimation
Channel delay $\tau$ exists between Tx and Rx.

For AWGN + channel delay $\tau$ + carrier phase mismatch $\phi_0$, given $s(t)$ transmitted, one receives

$$r(t) = s(t - \tau) + n(t).$$

As lowpass equivalent

$$s(t) = \text{Re} \left[ s_\ell(t) e^{i 2\pi f_c t} \right]$$
but

$$r(t) = \text{Re} \left[ r_\ell(t) e^{i 2\pi f_c t + i \phi} \right]$$

we have

$$r(t) = \text{Re} \left[ r_\ell(t) e^{i 2\pi f_c t + i \phi_0} \right]$$

$$= \text{Re} \left[ s_\ell(t - \tau) e^{i 2\pi f_c (t - \tau)} \right] + \text{Re} \left[ n_\ell(t) e^{i 2\pi f_c t + i \phi_0} \right]$$

$$= \text{Re} \left\{ \left[ s_\ell(t - \tau) e^{i \phi} + n_\ell(t) \right] e^{i 2\pi f_c t + i \phi_0} \right\}$$

where $\phi = -2\pi f_c \tau - \phi_0$. 
\[ \phi = -2\pi f_c \tau - \phi_0 \]

In general, \( \tau \ll T \), but

\[ |-2\pi f_c \tau| \mod 2\pi \]

is far from 0 since \( f_c \) is large.

So, we should treat \( \tau \) and \( \phi \) as different random variables

\[ r_\ell(t) = s_\ell(t; \phi, \tau) + n_\ell(t). \]

Note that since the passband signal \( s(t - \tau) \) is “real”, there does not appear a phase mismatch “\( \phi \)” for the real passband signals.

In other words, \( \phi \) appears due to the imperfect down-conversion at the receiver.
\[ r_\ell(t) = s_\ell(t; \phi, \tau) + n_\ell(t) \]

Let \( \Theta = (\phi, \tau) \), and set

\[ r_\ell(t) = s_\ell(t; \Theta) + n_\ell(t). \]

Let \( \{\phi_{n,\ell}(t), 1 \leq n \leq N\} \) be a set of orthonormal functions over \([0, T_0)\), where \( T_0 \geq T \), such that \( r_{j,\ell} = \langle r_\ell(t), \phi_{n,\ell}(t) \rangle \) and we have a vector representation

\[ r_\ell = s_\ell(\Theta) + n_\ell. \]
Assuming \( \Theta \) has a joint pdf \( f(\Theta) \), the MAP estimate of \( \Theta \) is

\[
\hat{\Theta} = \arg \max_{\Theta} f(\Theta| r_\ell) \\
= \arg \max_{\Theta} f(r_\ell| \Theta) \frac{f(\Theta)}{f(r_\ell)} = \arg \max_{\Theta} f(r_\ell| \Theta) f(\Theta).
\]

Assume \( \Theta \) is uniform and holds constant for an observation period of \( T_0 \geq T \) (slow variation),

\[
\hat{\Theta} = \arg \max_{\Theta} f(r_\ell| \Theta) f(\Theta) = \arg \max_{\Theta} f(r_\ell| \Theta)
\]

The latter is the **ML estimate of \( \Theta \)**.

Note that \( f(r_\ell| \Theta) \) is the likelihood function.
\[ r_\ell = s_\ell(\Theta) + n_\ell \quad \text{and} \quad \hat{\Theta} = \arg\max_{\Theta} f(r_\ell|\Theta) \]

For a fixed \( \{\phi_{j,\ell}(t)\}_{j=1}^N \) and \( E[|n_{j,\ell}|^2] = \sigma_{\ell}^2 = 2N_0 \) (cf. slides Chapter 4-11 and 4-116),

\[
f(r_\ell|\Theta) = \left( \frac{1}{\pi\sigma_{\ell}^2} \right)^N \exp \left\{ -\sum_{n=1}^N \frac{|r_{n,\ell} - s_{n,\ell}(\Theta)|^2}{\sigma_{\ell}^2} \right\}.
\]

Text (5.1-5) uses bandpass analysis and yields

\[
f(r|\Theta) = \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^N \exp \left\{ -\sum_{n=1}^N \frac{|r_n - s_n(\Theta)|^2}{2\sigma^2} \right\} \quad \text{with} \quad \sigma^2 = \frac{N_0}{2}.
\]

We will show later that both analyses yields the same result!

Assume that \( \{\phi_{n,\ell}(t)\}_{j=1}^N \) is a complete orthonormal basis. Then

\[
\sum_{n=1}^N |r_{n,\ell} - s_{n,\ell}(\Theta)|^2 = \int_0^{T_0} |r_\ell(t) - s_\ell(t; \Theta)|^2 \, dt.
\]
Given $s_\ell(t)$ known to both Tx and Rx, the ML estimate of $\Theta$ is

$$\hat{\Theta} = \arg \max_{\Theta} \Lambda(\Theta)$$

and the term

$$\Lambda(\Theta) = \exp \left\{ -\frac{1}{\sigma_\ell^2} \int_0^{T_0} |r_\ell(t) - s_\ell(t; \Theta)|^2 \, dt \right\}$$

will be referred to as the likelihood function.
FIGURE 5.1–2
Block diagram of an $M$-ary PSK receiver.
FIGURE 5.1-4
Block diagram of a QAM receiver.
5.2 Carrier phase estimation
Example 1 (DSB-SC signal where $\tau = 0$ and $\phi_0 = \tilde{\phi} - \phi$)

Assume the transmitted signal is

$$s(t) = A(t) \cos(2\pi f_c t + \phi)$$

Rx uses $c(t)$ with carrier reference $\tilde{\phi}$ to demodulate

$$c(t) = \cos(2\pi f_c t + \tilde{\phi})$$

So even $n_\ell(t) = 0$, the down-covertion Rx gives

$$LPF\{s(t)c(t)\} = \frac{1}{2}A(t)\cos(\phi - \tilde{\phi})$$

Performance is severely degraded due to phase error $(\phi - \tilde{\phi})$.

Hence, signal power is reduced by a factor $\cos^2(\phi - \tilde{\phi})$:

- A phase error of $10^\circ$ leads to 0.13 dB of signal power loss.
- A phase error of $30^\circ$ leads to 1.25 dB of signal power loss.
Example 2 (QAM or PSK where $\tau = 0$ and $\phi_0 = \tilde{\phi} - \phi$)

For QAM or PSK signals, $Tx$ sends

$$s(t) = x(t) \cos (2\pi f_c t + \phi) - y(t) \sin (2\pi f_c t + \phi)$$

and $Rx$ uses the $c_I(t)$ and $c_Q(t)$ to demodulate

$$c_I(t) = \cos (2\pi f_c t + \tilde{\phi}) \quad c_Q(t) = \sin (2\pi f_c t + \tilde{\phi})$$

Hence

$$r_I(t) = LPF\{s(t)c_I(t)\} = \frac{x(t) \cos(\phi - \tilde{\phi}) - y(t) \sin(\phi - \tilde{\phi})}{2}$$

$$r_Q(t) = LPF\{s(t)c_Q(t)\} = \frac{x(t) \sin(\phi - \tilde{\phi}) + y(t) \cos(\phi - \tilde{\phi})}{2}$$

Even worse, both power degradation and crosstalk occur.
5.2-1 ML carrier phase estimation
Assume $\tau = 0$ (or $\tau$ has been perfectly compensated) & estimate $\phi$.

So $\Theta = \phi$. The likelihood function is

$$\Lambda(\phi) = \exp \left\{ -\frac{1}{\sigma^2} \int_0^{T_0} |r_{\ell}(t) - s_{\ell}(t; \phi)|^2 \, dt \right\}$$

$$= \exp \left( -\frac{1}{\sigma^2} \int_0^{T_0} \left[ |r_{\ell}(t)|^2 - 2\text{Re} \{ r_{\ell}(t)s_{\ell}^*(t; \phi) \} + |s_{\ell}(t; \phi)|^2 \right] dt \right)$$

$|r_{\ell}(t)|^2$ is irrelevant to the maximization over $\phi$. 
For the term $|s_\ell(t; \phi)|^2$, we have

$$s_\ell(t; \phi) = s_\ell(t) e^{i\phi}.$$ 

So,

$$|s(t, \phi)|^2 = |s(t)|^2$$

Thus

$$\hat{\phi} = \arg \max_{\phi} \exp \left\{ \frac{2}{\sigma_\ell^2} \int_0^{T_0} \Re \{ r_\ell(t) s_\ell^*(t; \phi) \} \, dt \right\}$$

$$= \arg \max_{\phi} \exp \left\{ \frac{4}{\sigma_\ell^2} \int_0^{T_0} r(t) s(t; \phi) \, dt \right\}$$

(With $\sigma_\ell^2 = 2N_0$, this formula is the same as (5.2-8) obtained based on bandpass analysis!)

Note that from Slide 24 of Chapter 2,

$$\langle x(t), y(t) \rangle = \frac{1}{2} \Re \{ \langle x_\ell(t), y_\ell(t) \rangle \}.$$ 

We will use one of the two above criterions to derive $\hat{\phi}$, depending on whichever is convenient.
Assume

\[ s(t) = A \cos(2\pi f_c t) \quad \text{and} \quad r(t) = A \cos \left( 2\pi f_c t + \tilde{\phi} \right) + n(t) \]

where \( \tilde{\phi} \) is the unknown phase.

\[ \hat{\phi} = \arg \max_{\phi} \exp \left\{ \frac{4}{\sigma^2} \int_{0}^{T_0} r(t)s(t; \phi) \, dt \right\} \]

\[ = \arg \max_{\phi} \int_{0}^{T_0} r(t)s(t; \phi) \, dt \]
So we seek $\hat{\phi}$ that minimizes

$$\Lambda_L(\phi) = A \int_0^{T_0} r(t) \cos(2\pi f_c t + \phi) \, dt$$

Since $\phi$ is continuous, a necessary condition for a minimum is that

$$\left. \frac{d\Lambda_L(\phi)}{d\phi} \right|_{\phi=\hat{\phi}} = 0$$

It yields

$$\int_0^{T_0} r(t) \sin(2\pi f_c t + \hat{\phi}) \, dt = 0$$

$$= \cos(\hat{\phi}) \int_0^{T_0} r(t) \sin(2\pi f_c t) \, dt + \sin(\hat{\phi}) \int_0^{T_0} r(t) \cos(2\pi f_c t) \, dt$$

$$\hat{\phi} = -\tan^{-1} \frac{\int_0^{T_0} r(t) \sin(2\pi f_c t) \, dt}{\int_0^{T_0} r(t) \cos(2\pi f_c t) \, dt}$$
Recall

\[ r(t) = A \cos \left(2\pi f_c t + \tilde{\phi} \right) + n(t) \]

with unknown phase \( \tilde{\phi} \).

\[
\mathbb{E} \left[ \int_0^{T_0} r(t) \sin(2\pi f_c t) \, dt \right] = -\frac{1}{2} AT_0 \sin(\tilde{\phi})
\]

\[
\mathbb{E} \left[ \int_0^{T_0} r(t) \cos(2\pi f_c t) \, dt \right] = \frac{1}{2} AT_0 \cos(\tilde{\phi})
\]

Then on the average

\[
\hat{\phi} = -\tan^{-1} \left( \frac{\mathbb{E} \left[ \int_0^{T_0} r(t) \sin(2\pi f_c t) \, dt \right]}{\mathbb{E} \left[ \int_0^{T_0} r(t) \cos(2\pi f_c t) \, dt \right]} \right) = \tilde{\phi}
\]

which is the true phase.
A one-shot ML phase estimator
5.2-2 The phase-locked loops
Instead of one-shot estimate, a phase-locked loop continuously adjusts $\phi$ to achieve

$$\int_{0}^{T_0} r(t) \sin(2\pi f_c t + \hat{\phi}) \, dt = 0$$

We can then change the sign of sine function to facilitate the follow-up analysis.

$$\int_{0}^{T_0} r(t) (-\sin(2\pi f_c t + \hat{\phi})) \, dt = 0$$

**FIGURE 5.2–1**
A PLL for obtaining the ML estimate of the phase of an unmodulated carrier.
The analysis of the PLL can be visioned in a simplified basic diagram:

\[ r(t) = A \cos(2\pi f_c t + \phi) \]

\[ e(t) = \int_{T_0} (^{\ })dt \]

\[ -\sin(2\pi f_c t + \hat{\phi}_{ML}) \]

Tuning VCO to make it zero.
If $T_0$ is a multiple of $1/(2f_c)$, then

$$
\int_0^{T_0} e(t) dt = \frac{A}{2} \int_0^{T_0} \sin(\phi - \hat{\phi}_ML) dt - \frac{A}{2} \int_0^{T_0} \sin(4\pi f_c t + \phi + \hat{\phi}_ML) dt
$$

$$
= \frac{AT_0}{2} \sin(\phi - \hat{\phi}_ML) - 0
$$

- The effect of integration is similar to a lowpass filter.
\[ e(t) = \frac{A}{2} \sin(\phi - \hat{\phi}_{ML}) \]
\[ r(t) = A \cos(2\pi f_c t + \phi) \]
\[ \int_{T_0}^{T_0} e(t)dt = \frac{AT_0}{2} \sin(\phi - \hat{\phi}_{ML}) = 0 \]

\[ e(t) = \frac{A}{2} \sin(\phi - \hat{\phi}_{ML}) \]
\[ r(t) = A \cos(2\pi f_c t + \phi) \]
\[ \int_{T_0}^{T_0} e(t)dt = \frac{AT_0}{2} \sin(\phi - \hat{\phi}_{ML}) = 0 \]
Effective VCO output $\hat{\phi}(t)$ can be modeled as

$$\hat{\phi}(t) = K \int_{-\infty}^{t} v(\tau) d\tau$$

where $v(\cdot)$ is the input of the VCO.

\[ e(t) = \left( A/2 \right) \sin(\phi - \hat{\phi}_{ML}) \]
\[ r(t) = A \cos(2\pi f_c t + \phi) \]
\[ -\left( A/2 \right) \sin(4\pi f_c t + \phi + \hat{\phi}_{ML}) \]

VCO: $\hat{\phi} = K \int_{-\infty}^{t} v(\tau) d\tau$

Loop Filter $G(s)$

VCO =$K/s$
The nonlinear $\sin(\cdot)$ causes difficulty in analysis. Hence, we may simply it using $\sin(x) \approx x$.

In terms of Laplacian transform technique, we then derive the close-loop system transfer function.

$$H(s) = \frac{\hat{\phi}(s)}{\phi(s)} = \frac{\hat{\phi}(s)}{[\phi(s) - \hat{\phi}(s)] + \hat{\phi}(s)} = \frac{(KG(s)/s)[\phi(s) - \hat{\phi}(s)]}{[\phi(s) - \hat{\phi}(s)] + (KG(s)/s)[\phi(s) - \hat{\phi}(s)]} = \frac{KG(s)/s}{1 + KG(s)/s}$$
Second-order loop transfer function

\[ G(s) = \frac{1 + \tau_2 s}{1 + \tau_1 s}, \]  
where \( \tau_1 \gg \tau_2 \) for a lowpass filter.

\[ \Rightarrow H(s) = \frac{(2\zeta - \omega_n/K)(s/\omega_n) + 1}{(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1} \]

where

\[
\begin{align*}
\omega_n &= \sqrt{K/\tau_1} & \text{natural frequency of the loop} \\
\zeta &= \omega_n(\tau_2 + 1/K)/2 & \text{loop damping factor}
\end{align*}
\]
Assume $2\zeta - \frac{\omega_n}{K} = 2\zeta \left(1 - \frac{1}{K\tau_2+1}\right) \approx 2\zeta$. Hence,

$$H(s) \approx \frac{2\zeta (s/\omega_n) + 1}{(s/\omega_n)^2 + 2\zeta (s/\omega_n) + 1}.$$ 

- Damping factor=response speed (for changes)
**Noise-equivalent bandwidth of \( H(f) \)**

- **Definition:** One-sided noise-equivalent bandwidth of \( H(f) \)

\[
B_{eq} = \frac{1}{\max_f |H(f)|^2} \int_0^\infty |H(f)|^2 df = \frac{1 + (\frac{\tau_2 \omega_n}{2\zeta})^2}{8\zeta/\omega_n} \approx \frac{1 + 4\zeta^2}{8\zeta} \omega_n
\]

where we use \( \tau_2 \omega_n = 2\zeta - \frac{\omega_n}{K} \approx 2\zeta \).

- **Tradeoff in parameter selection in PLL**
  - It is desirable to have a larger PLL bandwidth \( \omega_n \) in order to track any time variation in the phase of the received carrier.
  - However, with a larger PLL bandwidth, more noise will be passed into the loop; hence, the phase estimate is less accurate.
5.2-3 Effect of additive noise on the phase estimate
Assume that

\[ r(t) = A_c \cos(2\pi f_c t + \phi(t)) + n(t) \]

where

\[ n(t) = n_c(t) \cos(2\pi f_c t + \phi(t)) - n_s(t) \sin(2\pi f_c t + \phi(t)) \]

and \( n_c(t) \) and \( n_s(t) \) are independent Gaussian random processes.
\[ e(t) = -r(t) \sin[2\pi f_c t + \hat{\phi}(t)] \]
\[ = -[A_c + n_c(t)] \cos(2\pi f_c t + \phi) \sin(2\pi f_c t + \hat{\phi}_{ML}) + n_s(t) \sin(2\pi f_c t + \phi) \sin(2\pi f_c t + \hat{\phi}_{ML}) \]
\[ = \frac{[A_c + n_c(t)]}{2} \sin(\phi - \hat{\phi}_{ML}) - \frac{[A_c + n_c(t)]}{2} \sin(4\pi f_c t + \phi + \hat{\phi}_{ML}) \]
\[ + \frac{n_s(t)}{2} \cos(\phi - \hat{\phi}_{ML}) - \frac{n_s(t)}{2} \cos(4\pi f_c t + \phi + \hat{\phi}_{ML}) \]

\[ r(t) = [A_c + n_c(t)] \cos[2\pi f_c t + \phi(t)] \]
\[ - n_s(t) \sin[2\pi f_c t + \phi(t)] \]
\[ \int_{T_0} e(t) dt = \frac{A_c}{2} \sin[\phi(t) - \hat{\phi}(t)] + \frac{n_c(t)}{2} \sin[\phi(t) - \hat{\phi}(t)] + \frac{n_s(t)}{2} \cos[\phi(t) - \hat{\phi}(t)] = 0 \]

\[ r(t) = [1 + n_c(t) / A_c] \cos[2\pi f_c t + \phi(t)] \]
\[ - [n_s(t) / A_c] \sin[2\pi f_c t + \phi(t)] \]
\[ e(t) = -r(t) \sin[2\pi f_c t + \hat{\phi}(t)] \]
\[ \int_{T_0} e(t) dt = \frac{1}{2} \sin[\phi(t) - \hat{\phi}(t)] + \frac{1}{2} n_2(t) = 0 \]

Let \( n_2(t) = \frac{1}{A_c} (n_c(t) \sin[\phi(t) - \hat{\phi}(t)] + n_s(t) \cos[\phi(t) - \hat{\phi}(t)]) \).
\[ r(t) = \left[ 1 + \frac{n_e(t)}{A_c} \right] \cos(2\pi f_c t + \phi(t)) - \left[ \frac{n_s(t)}{A_c} \right] \sin(2\pi f_c t + \phi(t)) \]

\[ e(t) = -r(t) \sin(2\pi f_c t + \hat{\phi}(t)) \]

\[ \frac{1}{2} \sin(\phi(t) - \hat{\phi}(t)) + \frac{1}{2} n_2(t) = 0 \]

\[ \frac{1}{T_0} \int_{T_0} \left( \right) dt \]
\[ r(t) = \left[ 1 + \frac{n_c(t)}{A_c} \right] \cos[2\pi f_c t + \phi(t)] - \left[ \frac{n_s(t)}{A_c} \right] \sin[2\pi f_c t + \phi(t)] \]

\[ e(t) = -r(t) \sin[2\pi f_c t + \phi(t)] = \frac{1}{2} v(t) \]

\[ = \frac{1}{2} \sin[\phi(t) - \hat{\phi}(t)] + \frac{1}{2} n_2(t) \]

\[ \| n_2(t) \text{ white with } \Phi_{n_2}(f) = \frac{N_0}{2A_c^2}. \]

**Linearized PLL model**
As a result,

\[
\frac{\hat{\phi}(s)}{\phi(s) + n_2(s)} = \frac{\hat{\phi}(s)}{[\phi(s) - \hat{\phi}(s) + n_2(s)] + \hat{\phi}(s)}
\]

\[
= \frac{(KG(s)/s)[(\phi(s) - \hat{\phi}(s)) + n_2(s)]}{[\phi(s) - \hat{\phi}(s) + n_2(s)] + (KG(s)/s)[(\phi(s) - \hat{\phi}(s)) + n_2(s)]}
\]

\[
= \frac{KG(s)/s}{1 + KG(s)/s} = H(s)
\]

\[
\Rightarrow \hat{\phi}(s) = [\phi(s) + n_2(s)]H(s)
\]

\[
\Rightarrow \hat{\phi}(t) = [\phi(t) + n_2(t)] \ast h(t) = \phi(t) \ast h(t) + n_2(t) \ast h(t)
\]

\[
\text{noise}
\]
Let’s calculate noise variance:

\[
\sigma^2_{\hat{\phi}} = \int_{-\infty}^{\infty} \Phi_{n_2}(f)|H(f)|^2 df
\]

\[
= \int_{-\infty}^{\infty} \frac{N_0}{2} \frac{1}{A_c^2} |H(f)|^2 df
\]

\[
= \frac{N_0 B_{eq}}{A_c^2} \max_f |H(f)|^2
\]

Define the SNR as

\[
\gamma_L = \frac{A_c^2}{N_0 \cdot B_{eq}}
\]

Then \(\sigma^2_{\hat{\phi}}\) is proportional to \(1/\gamma_L\).
Exact PLL model versus liberalized PLL model

**Exact PLL model**

\[ \phi(t) \rightarrow \phi(t) - \hat{\phi}(t) \rightarrow \sin(\cdot) \rightarrow n_2(t) \rightarrow \text{Loop Filter } G(s) \rightarrow v(t) \rightarrow VCO=K/s \]

**Linearized PLL model**

\[ \phi(t) \rightarrow \phi(t) - \hat{\phi}(t) \rightarrow \text{Loop Filter } G(s) \rightarrow v(t) \rightarrow VCO=K/s \]
It turns out that when \( G(s) = 1 \), the \( \sigma^2_\phi \) of the exact PLL model is tractable (Vitebi 1966).

He also derived \( f(\Delta \phi) = \frac{1}{2\pi I_0(\gamma L)} \exp\{\gamma L \cos(\Delta \phi)\} \).

The linearized model well approximates the exact model when \( \gamma_L > 3 \approx 4.77 \) dB.
5.2-4 Decision directed loops
For general modulation scheme, let

\[ s_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT); \]

then

\[ s(t; \phi) = \text{Re}\left\{ s_\ell(t) e^{i\phi} e^{i2\pi f_c t} \right\} \]

Hence, by letting \( T_0 = KT \),

\[ \Lambda_L(\phi) = \int_0^{T_0} r(t) s(t; \phi) \, dt \]

\[ = \int_0^{T_0} r(t) \text{Re}\left\{ \sum_{n=-\infty}^{\infty} I_n g(t - nT) e^{i\phi} e^{i2\pi f_c t} \right\} \, dt \]

\[ = \int_0^{T_0} r(t) \text{Re}\left\{ \sum_{n=0}^{K-1} I_n g(t - nT) e^{i\phi} e^{i2\pi f_c t} \right\} \, dt \]
\[ \Lambda_L(\phi) = \int_0^{T_0} r(t) \Re \left\{ \sum_{n=0}^{K-1} l_n g(t - nT) e^{i \phi} e^{i 2\pi f_c t} \right\} dt \]

\[
\Lambda_L(\phi) = \Re \left\{ e^{i \phi} \sum_{n=0}^{K-1} l_n \int_0^{T_0} r(t) g(t - nT) e^{i 2\pi f_c t} dt \right\} \\
= \Re \left\{ e^{i \phi} \sum_{n=0}^{K-1} l_n \int_{nT}^{(n+1)T} r(t) g(t - nT) e^{i 2\pi f_c t} dt \right\} \\
= \Re \left\{ e^{i \phi} \sum_{n=0}^{K-1} l_n y_n \right\} \\
= \Re \left\{ \sum_{n=0}^{K-1} l_n y_n \right\} \cos(\phi) - \text{Im} \left\{ \sum_{n=0}^{K-1} l_n y_n \right\} \sin(\phi) \]
\[ \Lambda_L(\phi) = \text{Re} \left\{ \sum_{n=0}^{K-1} l_n y_n \right\} \cos(\phi) - \text{Im} \left\{ \sum_{n=0}^{K-1} l_n y_n \right\} \sin(\phi) \]

Now

\[
\frac{d\Lambda_L(\phi)}{d\phi} = -\text{Re} \left\{ \sum_{n=0}^{K-1} l_n y_n \right\} \sin(\phi) - \text{Im} \left\{ \sum_{n=0}^{K-1} l_n y_n \right\} \cos(\phi)
\]

and the optimal estimate \( \hat{\phi} \) is given by

\[
\hat{\phi} = -\tan^{-1} \left( \frac{\text{Im} \left\{ \sum_{n=0}^{K-1} l_n y_n \right\}}{\text{Re} \left\{ \sum_{n=0}^{K-1} l_n y_n \right\}} \right)
\]

This is called decision directed estimation of \( \phi \).
Note that from Slide 24 of Chapter 2,

\[
\langle x(t), y(t) \rangle = \frac{1}{2} \text{Re} \{ \langle x_\ell(t), y_\ell(t) \rangle \}.
\]

Hence,

\[
\text{Re}\{ l_n y_n \} = \int_{nT}^{(n+1)T} r(t) \cdot \text{Re}\{ l_n g(t - nT) e^{i2\pi f_c t} \} \, dt
\]

\[
= \frac{1}{2} \text{Re} \left\{ \int_{nT}^{(n+1)T} r_\ell(t) l_n^* g^*(t - nT) \, dt \right\}
\]

\[
= \frac{1}{2} \text{Re} \left\{ l_n^* \int_{nT}^{(n+1)T} r_\ell(t) g^*(t - nT) \, dt \right\}
\]

\[
= \frac{1}{2} \text{Re} \{ l_n^* y_{n,\ell} \}
\]
\[
\text{Im}\{l_n y_n\} = \int_{nT}^{(n+1)T} r(t) \cdot \text{Im}\{l_n g(t - nT) e^{i 2\pi f_c t}\} \ dt
\]

\[
= \int_{nT}^{(n+1)T} r(t) \cdot \text{Re}\{(-i) l_n g(t - nT) e^{i 2\pi f_c t}\} \ dt
\]

\[
= \frac{1}{2} \text{Re}\left\{ \int_{nT}^{(n+1)T} r_\ell(t) \cdot i l_n^* g^*(t - nT) \ dt \right\}
\]

\[
= -\frac{1}{2} \text{Im}\{l_n^* y_n,\ell\}
\]

\[
\hat{\phi} = \tan^{-1}\left( \frac{\text{Im}\left\{ \sum_{n=0}^{K-1} l_n^* y_n,\ell \right\}}{\text{Re}\left\{ \sum_{n=0}^{K-1} l_n^* y_n,\ell \right\}} \right)
\]

**Final note:** The formula (5.2-38) in text has an extra “−” sign because the text (inconsistently to (5.1-2)) assumes

\[
s(t; \phi) = \text{Re}\left\{ s_\ell(t) e^{-i \phi} e^{i 2\pi f_c t}\right\}; \text{ but we assume}
\]

\[
s(t; \phi) = \text{Re}\left\{ s_\ell(t) e^{+i \phi} e^{i 2\pi f_c t}\right\}.
\]
FIGURE 5.2–9
Block diagram of double-sideband PAM signal receiver with decision-directed carrier phase estimation.
5.2-5 Non-decision-directed loops
For carrier phase estimation with $\sigma_\ell^2 = 2N_0$, we have shown that
\[
\hat{\phi} = \arg \max_{\phi} \exp \left\{ \frac{2}{N_0} \int_0^{T_0} r(t) s(t; \phi) \, dt \right\}
\]
\[
= -\tan^{-1} \left( \frac{\text{Im} \left\{ \sum_{n=0}^{K-1} I_n y_n \right\}}{\text{Re} \left\{ \sum_{n=0}^{K-1} I_n y_n \right\}} \right)
\]

When $\{I_n\}_{n=0}^{K-1}$ is unavailable, we take the expectation with respect to $\{I_n\}_{n=0}^{K-1}$ instead:
\[
\hat{\phi} = \arg \max_{\phi} \mathbb{E} \left[ \exp \left\{ \frac{2}{N_0} \int_0^{T_0} r(t) s(t; \phi) \, dt \right\} \right]
\]
\[
= \arg \max_{\phi} \mathbb{E} \left[ \exp \left\{ \frac{2}{N_0} \int_0^{T_0} r(t) \text{Re} \left\{ \sum_{n=0}^{K-1} I_n g(t - nT) e^{i\phi} e^{i2\pi f_c t} \right\} \, dt \right\} \right]
\]
\[
= \arg \max_{\phi} \mathbb{E} \left[ \exp \left\{ \frac{2}{N_0} \sum_{n=0}^{K-1} I_n y_n(\phi) \right\} \right]
\]

where we assume both $\{I_n\}$ and $g(t)$ are real and
\[
y_n(\phi) = \int_{nT}^{(n+1)T} r(t) g(t - nT) \cos(2\pi f_c t + \phi) \, dt.
\]
If \( \{l_n\} \) i.i.d. and equal-probable over \( \{-1, 1\} \),
\[
\hat{\phi} = \arg \max_{\phi} \prod_{n=0}^{K-1} \mathbb{E} \left[ \exp \left\{ \frac{2}{N_0} l_n y_n(\phi) \right\} \right] \\
= \arg \max_{\phi} \prod_{n=0}^{K-1} \left( \exp \left\{ -\frac{2}{N_0} y_n(\phi) \right\} + \exp \left\{ \frac{2}{N_0} y_n(\phi) \right\} \right) \\
= \arg \max_{\phi} \prod_{n=0}^{K-1} \cosh \left( \frac{2}{N_0} y_n(\phi) \right) \\
= \arg \max_{\phi} \sum_{n=0}^{K-1} \log \cosh \left( \frac{2}{N_0} y_n(\phi) \right)
\]

We may then determine the optimal \( \hat{\phi} \) by deriving
\[
\frac{\partial}{\partial \phi} \sum_{n=0}^{K-1} \log \cosh \left( \frac{2}{N_0} y_n(\phi) \right) = 0.
\]
For $|x| \ll 1$ (low SNR), $\log \cosh(x) \approx \frac{x^2}{2}$ (By Taylor expansion).

For $|x| \gg 1$ (high SNR), $\log \cosh(x) \approx |x|$.

$$
\hat{\phi} = \arg \max_{\phi} \sum_{n=0}^{K-1} \log \cosh \left( \frac{2}{N_0} y_n(\phi) \right)
$$

$$
= \begin{cases} 
\arg \max_{\phi} \sum_{n=0}^{K-1} \frac{2}{N_0^2} y_n^2(\phi) & N_0 \text{ large} \\
\arg \max_{\phi} \sum_{n=0}^{K-1} \frac{2}{N_0} |y_n(\phi)| & N_0 \text{ small} 
\end{cases}
$$

$$
= \begin{cases} 
\arg \max_{\phi} \sum_{n=0}^{K-1} y_n^2(\phi) & N_0 \text{ large} \\
\arg \max_{\phi} \sum_{n=0}^{K-1} |y_n(\phi)| & N_0 \text{ small} 
\end{cases}
$$
When \( x \) small,

\[
\log(\cosh(x)) = \log \frac{e^{-x} + e^x}{2} 
\]

\[
= \log \left[ 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + O(x^4) \right] + \left[ 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4) \right] 
\]

\[
= \log \left( 1 + \frac{1}{2}x^2 + O(x^4) \right) 
\]

\[
= \frac{1}{2}x^2 + O(x^4) 
\]

and

\[
\lim_{x \to \infty} \frac{\log(\cosh(x))}{x} = \lim_{x \to \infty} \tanh(x) = \lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = 1. 
\]
When $K = 1$, which covers the case of Example 5.2-2 in text, the optimal decision becomes irrelevant to $N_0$:

\[
\hat{\phi} = \begin{cases} 
\arg \max_{\phi} y_0^2(\phi) & N_0 \text{ large} \\
\arg \max_{\phi} |y_0(\phi)| & N_0 \text{ small}
\end{cases}
\]

\[
= \arg \max_{\phi} \left| \int_0^T r(t)g(t)\cos(2\pi f_c t + \phi) \, dt \right|
\]

\[
= \arg \max_{\phi} \left| \cos(\phi) \int_0^T r(t)g(t)\cos(2\pi f_c t) \, dt \right|
\]

\[
\quad - \sin(\phi) \int_0^T r(t)g(t)\sin(2\pi f_c t) \, dt \right|
\]

\[
= \arg \max_{\phi} \left| \cos(\phi) \cos(\theta) - \sin(\phi) \sin(\theta) \right| = \arg \max_{\phi} \left| \cos(\phi + \theta) \right|
\]

where $\tan(\theta) = \frac{\int_0^T r(t)g(t)\sin(2\pi f_c t) \, dt}{\int_0^T r(t)g(t)\cos(2\pi f_c t) \, dt}$. So the optimal $\hat{\phi}$ should make

\[
\hat{\phi} = -\theta = -\tan^{-1} \frac{\int_0^T r(t)g(t)\sin(2\pi f_c t) \, dt}{\int_0^T r(t)g(t)\cos(2\pi f_c t) \, dt}.
\]
5.3 Symbol timing estimation
Assume $\phi = 0$ (or $\phi$ has been perfectly compensated) & estimate $\tau$.

In such case,

$$r_\ell(t) = s_\ell(t; \tau) + n_\ell(t) = s_\ell(t - \tau) + n_\ell(t).$$

We could rewrite the likelihood function (cf. Slide 8 with $\sigma^2_\ell = 2N_0$) as

$$\Lambda(\tau) = \exp \left\{ -\frac{1}{2N_0} \int_0^{T_0} |r_\ell(t) - s_\ell(t; \tau)|^2 \, dt \right\}$$

$$= \exp \left( -\frac{1}{2N_0} \int_0^{T_0} \left[ |r_\ell(t)|^2 - 2\text{Re}\{r_\ell(t)s_\ell^*(t; \tau)\} + |s_\ell(t; \tau)|^2 \right] \, dt \right)$$

Same as before, the 1st term is independent of $\tau$ and can be ignored. But

$$\int_0^{T_0} |s_\ell(t; \tau)|^2 \, dt = \int_0^{T_0} |s_\ell(t - \tau)|^2 \, dt$$

could be a function of $\tau$.

So, we would “say” when $\tau \ll T$, the 3rd term is nearly independent of $\tau$ and can also be ignored.
This gives:

\[ \Lambda_L(\tau) = \text{Re} \left\{ \int_0^{T_0} r_\ell(t) s_\ell^*(t; \tau) \, dt \right\} \]

Assume

\[ s_\ell(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT) \]

Then

\[ \Lambda_L(\tau) = \text{Re} \left\{ \sum_{n=0}^{K-1} I_n^* \int_0^{T_0} r_\ell(t) g^*(t - nT - \tau) \, dt \right\} \]

\[ = \text{Re} \left\{ \sum_{n=0}^{K-1} I_n^* \tilde{y}_{n,\ell}(\tau) \right\} \]

where

\[ \tilde{y}_{n,\ell}(\tau) = \int_0^{T_0} r_\ell(t) g^*(t - nT - \tau) \, dt \]
The text assumes that both \( \{ I_n \} \) and \( g(t) \) are real; hence \( r_\ell(t) \) can be made real by eliminating the complex part, and

\[
\Lambda_L(\tau) = \sum_{n=0}^{K-1} I_n y_{n,\ell}(\tau)
\]

where

\[
y_{n,\ell}(\tau) = \int_0^{T_0} \text{Re}\{r_\ell(t)\} g(t - nT - \tau) dt.
\]

Then the optimal decision is the \( \hat{\tau} \) such that

\[
\frac{d\Lambda_L(\tau)}{dT} = \sum_{n=0}^{K-1} I_n \frac{dy_{n,\ell}(\tau)}{dT} = 0
\]

Likewise, it is called decision directed estimation.
Consider again the case of BPSK, i.e. $I_n = \pm 1$ equal-probable; then because the complex noise can be excluded, we have

$$\Lambda(\tau) = \exp \left( \frac{1}{N_0} \int_0^{T_0} \text{Re}\{r_{\ell}(t)\} s_{\ell}(t; \tau) dt \right)$$

$$= \exp \left( \frac{1}{N_0} \sum_{n=0}^{K-1} I_n \int_0^{T_0} \text{Re}\{r_{\ell}(t)\} g(t - nT - \tau) dt \right)$$

$$= \prod_{n=0}^{K-1} \exp \left( \frac{1}{N_0} I_n y_{n,\ell}(\tau) \right)$$

$$\bar{\Lambda}(\tau) = \mathbb{E} [\Lambda(\tau)] = \prod_{n=0}^{K-1} \cosh \left( \frac{1}{N_0} y_{n,\ell}(\tau) \right)$$
Thus

$$\tilde{\Lambda}_L(\tau) = \log \tilde{\Lambda}(\tau) = \sum_{n=0}^{K-1} \log \cosh \left( \frac{1}{N_0} y_{n,\ell}(\tau) \right)$$

For moderate SNR, people simplify $\log \cosh(x)$ to $\frac{1}{2} x^2$,

$$\tilde{\Lambda}_L(\tau) \approx \frac{1}{2N_0^2} \sum_{n=0}^{K-1} y_{n,\ell}^2(\tau).$$

Taking derivative, we see a necessary condition for $\hat{\tau}$ is

$$\left. \frac{d\tilde{\Lambda}_L(\tau)}{d\tau} \right|_{\tau=\hat{\tau}} = \sum_{n=0}^{K-1} y_{n,\ell}(\hat{\tau}) \frac{y'_{n,\ell}(\hat{\tau})}{y_{n,\ell}(\hat{\tau})} = 0$$
For high SNR, people simplify $\log \cosh(x)$ to $|x|$, 

$$\bar{\Lambda}_L(\tau) \approx \frac{1}{N_0} \sum_{n=0}^{K-1} |y_{n,\ell}(\tau)|.$$

Taking derivative, we see a necessary condition for $\hat{\tau}$ is

$$\left. \frac{d\bar{\Lambda}_L(\tau)}{d\tau} \right|_{\tau=\hat{\tau}} = \sum_{n=0}^{K-1} \text{sgn}(y_{n,\ell}(\hat{\tau})) y'_{n,\ell}(\hat{\tau}) = \sum_{n=0}^{K-1} |y_{n,\ell}(\hat{\tau})| \frac{y'_{n,\ell}(\hat{\tau})}{y_{n,\ell}(\hat{\tau})} = 0$$

Note that here, we use

$$\frac{\partial |f(x)|}{\partial x} = \begin{cases} f'(x), & f(x) > 0 \\ -f'(x), & f(x) < 0 \end{cases} = \text{sgn}(f(x)) f'(x).$$
5.4 Joint estimation of carrier phase and symbol timing
The likelihood function is

\[
\Lambda(\phi, \tau) = \exp\left\{ -\frac{1}{2N_0} \int_0^{T_0} |r_\ell(t) - s_\ell(t; \tau, \phi)|^2 \, dt \right\}
\]

Assuming

\[
s_\ell(t) = \sum_{n=-\infty}^{\infty} \left( I_n g(t - nT - \tau) + \imath J_n w(t - nT - \tau) \right)
\]

we have

\[
s_\ell(t; \phi, \tau) = \sum_{n=-\infty}^{\infty} \left( I_n g(t - nT - \tau) + \imath J_n w(t - nT - \tau) \right) e^{-\imath \phi}
\]

Here, I use \( e^{-\imath \phi} \) in order to “synchronize” with the textbook.

- **PAM:** \( I_n \) real and \( J_n = 0 \)
- **QAM and PSK:** \( I_n \) complex and \( J_n = 0 \)
- **OQPSK:** \( w(t) = g(t - T/2) \)
Along the same lines as before, we could rewrite $\Lambda(\phi, \tau)$ as

$$\Lambda_L(\phi, \tau)$$

$$= \text{Re} \left\{ \int_0^{T_0} r_\ell(t) s_\ell^*(t; \phi, \tau) \, dt \right\}$$

$$= \text{Re} \left\{ e^{i\phi} \sum_{n=0}^{K-1} \int_0^{T_0} r_\ell(t) \left( I_n^* g^* (t - nT - \tau) - \iota J_n^* w^* (t - nT - \tau) \right) \, dt \right\}$$

$$= \text{Re} \left\{ e^{i\phi} \sum_{n=0}^{K-1} \left( I_n^* y_{n,\ell}(\tau) - \iota J_n^* x_{n,\ell}(\tau) \right) \right\}$$

$$= \text{Re} \left\{ e^{i\phi} (A(\tau) + \iota B(\tau)) \right\} = A(\tau) \cos(\phi) - B(\tau) \sin(\phi)$$

where

$$\begin{align*}
y_{n,\ell}(\tau) &= \int_0^{T_0} r_\ell(t) g^* (t - nT - \tau) \, dt \\
x_{n,\ell}(\tau) &= \int_0^{T_0} r_\ell(t) w^* (t - nT - \tau) \, dt \\
A(\tau) + \iota B(\tau) &= \sum_{n=0}^{K-1} \left( I_n^* y_{n,\ell}(\tau) - \iota J_n^* x_{n,\ell}(\tau) \right)
\end{align*}$$
The necessary conditions for \( \hat{\phi} \) and \( \hat{\tau} \) are

\[
\left. \frac{\partial \Lambda_L(\phi, \tau)}{\partial \tau} \right|_{\tau = \hat{\tau}} = 0 \quad \text{and} \quad \left. \frac{\partial \Lambda_L(\phi, \tau)}{\partial \phi} \right|_{\phi = \hat{\phi}} = 0
\]

Finally we have the optimal estimates given by

\[
\hat{\tau} \text{ satisfies } A(\tau) \frac{\partial A(\tau)}{\partial \tau} + B(\tau) \frac{\partial B(\tau)}{\partial \tau} = 0
\]

\[
\hat{\phi} = - \tan^{-1} \frac{B(\hat{\tau})}{A(\hat{\tau})}
\]
5.5 Performance characteristics of ML estimators
Comparison between decision-directed (DD) and non-decision-directed (NDD) estimators

Comparison between symbol timing (i.e., $\tau$) DD and NDD estimates with raised-cosine(-spectrum) pulse shape.

- $\beta$ is a parameter of the raised-cosine pulse
Raise-cosine spectrum

\[ X_{rc}(f) = \begin{cases} 
T, & 0 \leq |f| \leq \frac{1-\beta}{2T}; \\
\frac{T}{2} \left\{ 1 + \cos \left[ \frac{\pi T}{\beta} \left( |f| - \frac{1-\beta}{2T} \right) \right] \right\}, & \frac{1-\beta}{2T} \leq |f| \leq \frac{1+\beta}{2T}; \\
0, & \text{otherwise}
\]
Raise-cosine spectrum

The diagrams illustrate the raise-cosine spectrum for different values of \( \beta \):

- \( \beta = 0 \):
  - Upper diagram: \( x(t) \)
  - Lower diagram: \( X(f) \)

- \( \beta = 0.5 \)
  - Upper diagram: \( x(t) \)
  - Lower diagram: \( X(f) \)

- \( \beta = 1 \)
  - Upper diagram: \( x(t) \)
  - Lower diagram: \( X(f) \)
Comparison between DD and NDD estimates with raised-cosine(-spectrum) pulse shape.

- The larger the excess bandwidth, the better the estimate.
- NDD variance may go without bound when $\beta$ small.
What you learn from Chapter 5

- MAP/ML estimate of $\tau$ and $\phi$ based on likelihood ratio function and known signals
- Phase lock loop
  - Linearized model analysis and its transfer function with and without additive noise
  - (Good to know) Noise-equivalent bandwidth
- Decision-directed (or decision-feedback) loop
- Non-decision-directed loop
  - Take expectation on probability, not on a function of probability such as Euclidean distance