In Chapter 9: *Communications Through Band-Limited Channels*, we have seen that when the channel is band-limited to $[-W, W]$, without extra care, the received signal at matched-filter output is

$$y_k = \sum_{n=-\infty}^{\infty} I_n x_{k-n} + z_k$$

where

- $I_k$ is the information symbol,
- $x_k$ is the overall discrete impulse response,
- $z_k$ is the additive noise

This gives an intersymbol interference (ISI) channel.
Motivation

\[ y_k = \sum_{n=-\infty}^{\infty} I_n x_{k-n} + z_k \]

With Nyquist pulse, it is possible to create an ISI-free channel

\[ x_{k-n} = \delta_{k-n} \quad \text{and} \quad y_k = I_k + z_k. \]

However, due to

- mis-synchronization
- imperfect channel estimation, etc

ISI is sometimes inevitable.
There is another simple solution to the ISI problem. The main idea is the following:

- Given the discrete channel impulse response $x_k$, we see

$$y_k = I_k * x_k + z_k$$

- By Fourier duality, taking discrete Fourier transform (DFT) at both sides gives

$$Y_k = \mathcal{F}_k X_k + Z_k$$

This transforms “convolution” to “multiplication.”
So, if we set, for example, \( \{I_k\} \in \{-1, 1\} \) and transmit its IDFT \( \{l_k\} \). Then, the ISI problem can be solved straightforwardly.

Note that \( Y_k \) is only a function of \( I_k \) and does not depend on \( \ldots, I_{k-2}, I_{k-1}, I_{k+1}, I_{k+2}, \ldots \).

This idea has been employed in many modern techniques such as **Orthogonal Frequency Division Multiplexing (OFDM)**.
11.2 Multicarrier communications:

11.2-3 Orthogonal frequency division multiplexing (OFDM)

11.2-4 Modulation and demodulation in an OFDM system
Let $T$ be the symbol duration; then we know the set of waveforms

$$\left\{ \kappa e^{\imath 2\pi \frac{k}{T} t} : t \in [0, T), k = 0, 1, \ldots, Q - 1 \right\}$$

is a set of orthonormal functions, where

$$\kappa = \sqrt{\frac{1}{T}}.$$

$$\begin{aligned}
\left\langle \kappa e^{\imath 2\pi \frac{k}{T} t}, \kappa e^{\imath 2\pi \frac{j}{T} t} \right\rangle &= \int_0^T \kappa^2 e^{\imath 2\pi \frac{k}{T} t} e^{-\imath 2\pi \frac{j}{T} t} dt \\
&= \frac{1}{T} \int_0^T e^{\imath 2\pi \frac{(k-j)}{T} t} dt \\
&= \delta_{k-j}
\end{aligned}$$
Let

\[ X_{k,n} = I_{k,n} + \imath Q_{k,n} \]

be the QAM symbol at the \( k \)th subcarrier and at the \( n \)th symbol period; then the multicarrier waveform is given by

\[ s_\ell(t) = \kappa \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{Q-1} X_{k,n} e^{\imath 2\pi \frac{k}{T} t} \right) g(t - nT) \]

where \( g(t) \) is the pulse shaping function.

Hence,

\[ s(t) = \text{Re} \left\{ s_\ell(t) e^{\imath 2\pi f_c t} \right\} \]

At the first glance, it seems to be a single-carrier \( f_c \) system; but, it is actually a multi-carrier system with single-carrier implementation.
\[ s(t) = \Re \{ s_\ell(t) e^{i2\pi f_c t} \} \]

\[ = \Re \left\{ \kappa \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{Q-1} X_{k,n} e^{i\frac{2\pi k}{T} t} \right) g(t - nT) e^{i2\pi f_c t} \right\} \]

\[ = \sum_{k=0}^{Q-1} \Re \left\{ \left( \kappa \sum_{n=-\infty}^{\infty} X_{k,n} g(t - nT) \right) e^{i2\pi f_k t} \right\} \]

where \( f_k = f_c + \frac{k}{T} \) is the \( k \)th carrier.
11.2-6 Spectral characteristics of multicarrier signals
Clearly, $s_\ell(t)$ is a random process.

For simplicity, we may assume $I_{k,n}$ and $Q_{k,n}$ are i.i.d., zero mean, and variance $\frac{1}{2}\sigma^2$.

With $\kappa = 1/\sqrt{T}$, the autocorrelation function of $s_\ell(t)$ is

\[
R_{s_\ell}(t + \tau, t) = \frac{1}{T} \mathbb{E} \left[ \left( \sum_{n=\infty}^{Q-1} \sum_{k=0}^{\infty} X_{k,n} g(t + \tau - nT) e^{i 2\pi \frac{k}{T}(t+\tau)} \right) \left( \sum_{m=\infty}^{Q-1} \sum_{j=0}^{\infty} X_{j,m}^* g^*(t - mT) e^{-i 2\pi \frac{j}{T} t} \right) \right]
\]

\[
= \frac{1}{T} \sigma^2 \sum_{k=0}^{Q-1} \sum_{n=\infty}^{\infty} g(t + \tau - nT) g^*(t - nT)
\]
It is clear that

\[ R_{s_\ell}(t + \tau, t) = R_{s_\ell}(t + \tau + mT, t + mT) \]

for any integer \( m \); hence \( s_\ell(t) \) is a cyclostationary random process with period \( T \).

The average autocorrelation function is thus given by

\[ \bar{R}_{s_\ell}(\tau) = \frac{1}{T} \int_0^T R_{s_\ell}(t + \tau, t) \, dt \]

\[ = \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} e^{i2\pi k\tau\frac{T}{T}} \sum_{n=-\infty}^{\infty} \int_0^T g(t + \tau - nT) g^*(t - nT) \, dt \]

\[ = \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} e^{i2\pi k\tau\frac{T}{T}} \sum_{n=-\infty}^{\infty} \int_{-nT}^{-(n-1)T} g(u + \tau) g^*(u) \, du \]

\[ = \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} e^{i2\pi k\tau\frac{T}{T}} \int_{-\infty}^{\infty} g(t + \tau) g^*(t) \, dt \]
The time-average power spectral density of $s_\ell(t)$ is

$$\bar{S}_{s_\ell}(f) = \int_{-\infty}^{\infty} \bar{R}_{s_\ell}(\tau) e^{-i 2\pi f \tau} \, d\tau$$

$$= \sigma^2 \frac{Q-1}{T^2} \sum_{k=0}^{Q-1} \int_{-\infty}^{\infty} g^*(t) \left( \int_{-\infty}^{\infty} g(t+\tau) e^{-i 2\pi \left( f - \frac{k}{T} \right) \tau} \, d\tau \right) \, dt$$

$$= \sigma^2 \frac{Q-1}{T^2} \sum_{k=0}^{Q-1} \int_{-\infty}^{\infty} g^*(t) \left( \int_{-\infty}^{\infty} g(u) e^{-i 2\pi \left( f - \frac{k}{T} \right) u} \, du \right) e^{i 2\pi \left( f - \frac{k}{T} \right) t} \, dt$$

$$= \sigma^2 \frac{Q-1}{T^2} \sum_{k=0}^{Q-1} G \left( f - \frac{k}{T} \right) \left( \int_{-\infty}^{\infty} g(t) e^{-i 2\pi \left( f - \frac{k}{T} \right) t} \, dt \right)^*$$

$$= \sigma^2 \frac{Q-1}{T^2} \sum_{k=0}^{Q-1} \left| G \left( f - \frac{k}{T} \right) \right|^2.$$
Theorem 1

The time-average power spectral density of $s_\ell(t)$ is

$$\bar{S}_{s_\ell}(f) = \frac{\sigma^2}{T^2} \sum_{k=0}^{Q-1} \left|G \left( f - \frac{k}{T} \right) \right|^2$$

where $Q$ is the number of subcarriers.
Example

Let \( g(t) \) be the rectangular pulse shape of height 1 and duration \( T \); then

\[
G(f) = e^{-i\pi f T} T \text{sinc}(fT).
\]

Hence

\[
\tilde{S}_{s_\ell}(f) = \sigma^2 \sum_{k=0}^{Q-1} \left| \text{sinc}\left(\left(f - \frac{k}{T}\right)T\right)\right|^2.
\]

In particular,

\[
\tilde{S}_{s_\ell}\left(\frac{m}{T}\right) = \sigma^2 \sum_{k=0}^{Q-1} \left| \text{sinc}(m - k) \right|^2 = \begin{cases} 
\sigma^2, & \text{if } 0 \leq m < Q \\
0, & \text{otherwise.}
\end{cases}
\]
Example: $T = 1$ and $Q = 5$

Figure: $\left| G \left( f - \frac{k}{T} \right) \right|^2$ for $k = 0, 1, 2, 3, 4$
Example: $T = 1$ and $Q = 5$

Figure: $S_{s_{\ell}}(f)$
\[
\bar{S}_{s_\ell}(f) = \sigma^2 \sum_{k=0}^{Q-1} \left| \text{sinc} \left( (f - \frac{k}{T}) T \right) \right|^2
\]

- The PSD $\bar{S}_{s_\ell}(f)$ decays very slow at high frequencies at rate approximately

\[
\bar{S}_{s_\ell}(f) \approx \frac{1}{f^2}.
\]

- Out of band power leakage is severe and the resulting spectrum may not meet the FCC requirement.

- One can add a bandpass filter afterwards to remove the out-of-band signals, for example, using the root raised cosine filters.
11.2-5 An FFT algorithm implementation of an OFDM system
For simplicity, we again assume \( g(t) \) is the rectangular pulse shape of height 1 and duration \( T \) such that for \( 0 \leq t < T \),

\[
s_\ell(t) = \kappa \sum_{k=0}^{Q-1} X_k e^{j2\pi \frac{k}{T} t}
\]

and zero, otherwise, where we drop the subscript \( n \) for symbol period for notational convenience.

Then, we will introduce an efficient way to generate the following waveform:

\[
s_\ell(t) = \kappa \sum_{k=0}^{Q-1} X_k e^{j2\pi \frac{k}{T} t}, \quad t \in [0, T)
\]
Generating \( s_\ell(t) \) using iFFT + DAC

- Consider an \( N \)-point iFFT with \( N \geq Q \). (Usually, \( N \) is equal to the power of two.)

- Set \( \hat{X}_k = \begin{cases} X_k, & \text{if } 0 \leq k < Q \\ 0, & \text{if } Q \leq k < N \end{cases} \)

- The iFFT of \( \hat{X}_k \) is given by

\[
\hat{x}_m = \frac{1}{N} \sum_{k=0}^{N-1} \hat{X}_k e^{i 2\pi \frac{mk}{N}} = \frac{1}{N} \sum_{k=0}^{Q-1} X_k e^{i 2\pi \frac{mk}{N}}
\]

- Feeding \( N\hat{x}_m \) to a digital-analog-converter (DAC) at rate \( \frac{N}{T} \) gives

\[
\hat{s}_\ell(t) = (\kappa N) \sum_{m=0}^{N-1} \hat{x}_m \ g_{DAC} \left( t - \frac{m}{N} T \right)
\]

where \( g_{DAC}(t) \) is the rectangular pulse of height 1 and duration \( \frac{T}{N} \).
Note that for \( n = 0, 1, \ldots, N - 1 \),

\[
\hat{s}_\ell \left( \frac{n}{N} T \right) = \kappa N \sum_{m=0}^{N-1} \hat{x}_m \ g_{\text{DAC}} \left( \frac{n}{N} T - \frac{m}{N} T \right) \\
= \kappa N \hat{x}_n \\
= \kappa \sum_{k=0}^{Q-1} X_k e^{i 2\pi \frac{kn}{N}} = s_\ell \left( \frac{n}{N} T \right) \quad \text{(See Slide 11-20.)}
\]

We see

\[
\hat{s}_\ell(t) = s_\ell(t) \quad \text{for} \quad t = nT/N \quad \text{and} \quad n = 0, 1, \ldots, N - 1.
\]

The technique we had used is called **Zero Padding** in DSP.
Example: $Q = 16$ and $T = 1$

$$X_k = I_k + \tau Q_k \text{ for } 0 \leq k < Q = 16$$
Example: $Q = 16$ and $T = 1$

Figure: $s_\ell(t) = I(t) + \imath Q(t) = \kappa \sum_{k=0}^{Q-1} X_k e^{\imath 2\pi \frac{k}{T} t}, \quad t \in [0, T)$
Example: \( Q = 16 \) and \( T = 1 \) and \( N = 16 \)

\[
s_{\ell}(t) = \hat{s}_{\ell}(t) \quad \text{at} \quad t = 0, \frac{1}{16}, \frac{2}{16}, \ldots, \frac{15}{16} \quad \text{(sec)}
\]
Example: \( Q = 16 \) and \( T = 1 \) and \( N = 128 \)

\[
s_\ell(t) = \hat{s}_\ell(t) \quad \text{at} \quad t = 0, \frac{1}{128}, \frac{2}{128}, \ldots, \frac{127}{128} \text{ (sec)}
\]
Example: $Q = 16$ and $T = 1$ and $N = 256$

$$s_\ell(t) = \hat{s}_\ell(t) \text{ at } t = \frac{1}{256}, \frac{2}{256}, \ldots, \frac{255}{256} \text{ (sec)}$$
Example: \( Q = 16 \) and \( T = 1 \)

\[
\begin{align*}
s_{\ell}(t) &= \kappa \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{k}{T} t} g(t), \quad t \in [0, T) \\
S_{\ell}(f) &= \mathcal{F}\{s_{\ell}(t)\} = I(f) + \iota Q(f) = \kappa \sum_{k=0}^{Q-1} X_k G\left(f - \frac{k}{T}\right)
\end{align*}
\]

Out-of-band leakage due to rectangular pulse shape \( g(t) \)
Example: \( Q = 16 \) and \( T = 1 \)

\[
S_\ell \left( f = \frac{k}{T} \right) = X_k = l_k + \kappa Q_k \text{ for } k = 0, 1, \ldots, Q = 15 \text{ and } \kappa = \frac{1}{T}
\]
Transmission of multicarrier signal
\[ \hat{s}_\ell(t) = (\kappa N) \sum_{n=-\infty}^{\infty} \left( \sum_{m=0}^{N-1} \hat{x}_{m,n} g_{DAC} \left(t - \frac{m}{N} T\right)\right) g(t - nT) \]

\[ = \kappa \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{Q-1} X_{k,n} e^{i2\pi \frac{mk}{N}} \right) g_{DAC} \left(t - \frac{m}{N} T\right) g(t - nT) \]

\[ = \kappa \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{Q-1} X_{k,n} \left( \sum_{m=0}^{N-1} e^{i2\pi \frac{mk}{N}} \right) g_{DAC} \left(t - \frac{m}{N} T\right) \right) g(t - nT) \]

\[ = \kappa \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{Q-1} X_{k,n} e^{i2\pi \frac{k}{T} \frac{t(N/T)}{(N/T)}} \right) g(t - nT) \]

\[ s_\ell(t) = \kappa \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{Q-1} X_{k,n} e^{i2\pi \frac{k}{T} t} \right) g(t - nT) \]

The difference between ideal \( s_\ell(t) \) and physically realizable \( \hat{s}_\ell(t) \) is that the latter uses a “digitized” time scale.
Denote $a = \frac{\lfloor t(N/T) \rfloor}{t(N/T)}$, which is approximately 1 when $N$ large.

Then the transmitted signal is given by

\[
\hat{s}(t) = \text{Re}\left\{ \hat{s}_\ell(t) e^{j2\pi f_c t} \right\}
\]

\[
= \kappa \sum_{n=-\infty}^{\infty} \text{Re}\left\{ \left( \sum_{k=0}^{Q-1} X_{k,n} e^{j2\pi \frac{k}{T} \frac{t(N/T)}{(N/T)}} \right) e^{j2\pi f_c t} \right\} g(t - nT)
\]

\[
= \kappa \sum_{n=-\infty}^{\infty} \sum_{k=0}^{Q-1} \left\{ l_{k,n} \cos \left[ 2\pi \left( f_c + a \frac{k}{T} \right) t \right] 
\right. \\
\quad \left. - Q_{k,n} \sin \left[ 2\pi \left( f_c + a \frac{k}{T} \right) t \right] \right\} g(t - nT)
\]
Transmission of multicarrier signal
OFDM = Multicarrier + Cyclic prefix

- Why adding cyclic prefix?
  To combat the channel effect due to $c_\ell(t)$.

- We can virtually think that
  \[
  s_\ell(t) = \begin{cases} 
  \kappa \sum_{k=0}^{Q-1} X_k e^{i 2\pi \frac{k}{T} t}, & t \in [0, T) \\
  0, & \text{otherwise}
  \end{cases}
  \]
  or more physically
  \[
  \hat{s}_\ell(t) = \begin{cases} 
  \kappa \sum_{k=0}^{Q-1} X_k e^{i 2\pi \frac{k}{T} a t}, & t \in [0, T) \\
  0, & \text{otherwise}
  \end{cases}
  \]

- Virtually extend $s_\ell(t)$ to make it periodic
  \[
  \tilde{s}_\ell(t) = \sum_{n=-\infty}^{\infty} s_\ell(t - nT) = \kappa \sum_{k=0}^{Q-1} X_k e^{i 2\pi \frac{k}{T} t} \quad \text{for } t \in \mathbb{R}
  \]
We will transmit $\tilde{s}_\ell(t)$ (of duration $P + T$) instead of $s_\ell(t)$ (of duration $T$) for OFDM, where $P$ is the length of $c_\ell(t)$.

In other words, we essentially assume that

$$c_\ell(t) = 0 \text{ for } t < 0 \text{ and } t \geq P.$$

The extra periodic part $P$ is called **cyclic prefix** in OFDM.

Usually, $T$ should be made much larger than $P$ in order to reduce the loss in transmission time and to save extra transmission power. For example, $T = 3.2\mu s$ and $P = 0.8\mu s$ for IEEE 802.11.

The necessity of adding CP will be clear in the analysis of Rx.
Receiver for multicarrier signal
Receiver for multicarrier signal

Oversampling

While there are only $Q$ tones transmitted, oversampling is required to avoid aliasing caused by out-of-band signals from other users.
Assuming the channel has a lowpass equivalent impulse response \( c_\ell(t) \), the received noise-free received signal is

\[
r_\ell(t) = \tilde{s}_\ell(t) \star c_\ell(t) = \int_0^P c_\ell(\tau)\tilde{s}_\ell(t-\tau)\,d\tau,
\]

where \( \tilde{s}_\ell(t) \) periodic with period \( T \).

Since all we need is \( r_\ell(t) \) for \( t \in [0, T) \), it is clear from the above formula that we only need \( \tilde{s}_\ell(t) \) for \( t \in [-P, T) \).

By this CP technique, the received signal is simplified to:

\[
r_\ell(t) = \tilde{s}_\ell(t) \star c_\ell(t)
= \kappa \left( \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{k}{T} t} \right) \star c_\ell(t)
= \kappa \sum_{k=0}^{Q-1} X_k \int_{-\infty}^{\infty} c_\ell(\tau) e^{i2\pi \frac{k}{T} (t-\tau)} \,d\tau
\]
\[ r_\ell(t) = \kappa \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{k}{T} t} \int_{-\infty}^{\infty} c_\ell(\tau) e^{-i2\pi \frac{k}{T} \tau} d\tau \]

\[ = \kappa \sum_{k=0}^{Q-1} X_k e^{i2\pi \frac{k}{T} t} C_\ell \left( \frac{k}{T} \right). \]

Note \( r_\ell(t) \) is actually periodic with period \( T \).

Sample \( r_\ell(t) \) at rate \( \tilde{N} \frac{T}{\tilde{N}} \), where \( \tilde{N} \) is not necessarily equal to \( N \).

\[ r_m = r_\ell \left( \frac{m}{\tilde{N}} T \right) = \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{T} \right) X_k e^{i2\pi \frac{km}{\tilde{N}}} \]

* The extra receptions for \( m = -1, -2, \ldots, -\frac{P}{T} \tilde{N} \) due to CP are unused.
When using the physical $\hat{s}_\ell(t)$ instead of ideal $s_\ell(t)$,

$$
\hat{r}_m = \hat{r}_\ell \left( \frac{m}{\tilde{N}} T \right) = \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{T} \right) X_k e^{i 2\pi k \frac{m(N/\tilde{N})}{m(N/\tilde{N})} \frac{m}{\tilde{N}} T} 
$$

$$
= \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{T} \right) X_k e^{i 2\pi k \frac{m(N/\tilde{N})}{N}} \text{ for } 0 \leq m \leq \tilde{N} - 1
$$

So, if $N = \tilde{N}$ or $N$ is a multiple of $\tilde{N}$ (i.e., the sampling rate at Tx is higher), then $\hat{r}_m = r_m$.

However, if $\tilde{N}$ is a multiple of $N$, say, $\tilde{N} = uN$, then

$$
\hat{r}_m = \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{T} \right) X_k e^{i 2\pi k \frac{m(u)/N}{N}} = r_{u\lfloor m/u \rfloor}.
$$

In other words, we only have $N$ different samples at Rx since Tx only transmits $N$ samples.
The FFT/iFFT duality we adopt here is:

\[
\begin{align*}
\text{FFT} & \quad \hat{X}_k = \sum_{m=0}^{N-1} \hat{x}_m e^{-i 2\pi \frac{mk}{N}} \\
\text{iFFT} & \quad \hat{x}_m = \frac{1}{N} \sum_{k=0}^{N-1} \hat{X}_k e^{i 2\pi \frac{mk}{N}}
\end{align*}
\]
Channel equalization

Given the received signal vector \( r = [r_0, \cdots, r_{\tilde{N}-1}] \), the receiver applies FFT to \( r \) (Implicitly, \( N \) is a multiple of \( \tilde{N} \) with \( \tilde{N} > Q \).)

\[
R_n = \sum_{m=0}^{\tilde{N}-1} r_m e^{-i 2\pi \frac{mn}{\tilde{N}}}
\]

\[
= \sum_{m=0}^{\tilde{N}-1} \left( \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{T} \right) X_k e^{i 2\pi \frac{km}{\tilde{N}}} \right) e^{-i 2\pi \frac{mn}{\tilde{N}}}
\]

\[
= \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{T} \right) X_k \sum_{m=0}^{\tilde{N}-1} e^{-i 2\pi \frac{m(n-k)}{\tilde{N}}}
\]

\[
= \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{T} \right) X_k \cdot \tilde{N} \delta_{n-k}
\]

\[
= \begin{cases} 
\kappa \tilde{N} C_\ell \left( \frac{n}{T} \right) X_n, & 0 \leq n < Q \\
0, & Q \leq n < \tilde{N}
\end{cases}
\]
When oversampling occurs

When $\tilde{N} = uN$,

\[
R_n = \sum_{m=0}^{\tilde{N}-1} \hat{r}_m e^{-\mathbf{i}2\pi \frac{mn}{\tilde{N}}}
\]

\[
= \sum_{m=0}^{\tilde{N}-1} \left( \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{T} \right) X_k \left( \sum_{j=0}^{u-1} e^{-\mathbf{i}2\pi \frac{n_j}{u\tilde{N}}} \sum_{i=0}^{N-1} e^{-\mathbf{i}2\pi \frac{i(n-k)}{\tilde{N}}} \right) \right) e^{-\mathbf{i}2\pi \frac{mn}{\tilde{N}}} \quad (m = ui + j)
\]

\[
= \kappa \sum_{k=0}^{Q-1} C_\ell \left( \frac{k}{T} \right) X_k \left( \sum_{j=0}^{u-1} e^{-\mathbf{i}2\pi \frac{n_j}{u\tilde{N}}} \sum_{i=0}^{N-1} e^{-\mathbf{i}2\pi \frac{i(n-k)}{\tilde{N}}} \right)
\]

\[
= \begin{cases} 
\kappa \left( \sum_{j=0}^{u-1} e^{-\mathbf{i}2\pi \frac{n_j}{u\tilde{N}}} \right) NC_\ell \left( \frac{n \mod N}{T} \right) X_{n \mod N}, & 0 \leq n \mod N < Q \\
0, & Q \leq n \mod N < N 
\end{cases}
\]

\[
= \begin{cases} 
\kappa \frac{e^{-\mathbf{i}\pi \frac{(u-1)n}{u\tilde{N}}} \sin \left( \frac{\pi n}{\tilde{N}} \right)}{\sin \left( \frac{\pi n}{u\tilde{N}} \right)} NC_\ell \left( \frac{n \mod N}{T} \right) X_{n \mod N}, & 0 \leq n \mod N < Q \\
0, & Q \leq n \mod N < N 
\end{cases}
\]
Example. \( N = 16 \) and \( \tilde{N} = 64 \)

\[
\left| \frac{\sin\left(\frac{\pi n}{N}\right)}{\sin\left(\frac{\pi n}{uN}\right)} \right|
\]

\( n \)
With noise present, we have

\[ R_k = \kappa \tilde{N} C_\ell \left( \frac{k}{T} \right) X_k + Z_k \]

- Only one-tap equalization (i.e., \( \kappa \tilde{N} C_\ell \left( \frac{k}{T} \right) \)) is needed.
Disadvantages of OFDM

While OFDM allows for simple equalization, it also introduces other problems such as:

High peak-to-average power ratio (PAPR) at $s_{\ell}(t)$
What you learn from Chapter 11

- Spectral characteristics of multicarrier signals
- An FFT implementation of an OFDM system with DAC consideration
- Physical transmission of multicarrier signal over digitized time scale
- Multicarrier + Cyclic prefix
- Oversampling and undersampling at RX