Sample Problems for the Quiz on 3rd Nov

1. (a) Briefly describe MAP and ML criterion.
(b) Assume transmitted signals are

\[ \tilde{s}_m = [s_{m1}, s_{m2}, \ldots, s_{mM}] \quad m = 1, 2, \ldots, M \]

Received signal is

\[ \tilde{r} = [r_1, r_2, \ldots, r_K] = \tilde{s} + \tilde{n} \quad 1 \leq i \leq M \]

where

\[ \tilde{n} = [n_1, n_2, \ldots, n_K] \quad n_i \sim \mathcal{N}(0, \frac{N_0}{2}) \text{ and iid for } 1 \leq j \leq K \]

How to determine the decision rule using ML criterion?

2. (Problems 4.9 in the textbook)

A ternary communication system transmits one of three equiprobable signals \( s(t) \), 0, or \(-s(t)\) every \( T \) seconds. The received signal is \( r(t) = s(t) + z(t) \), \( r(t) = z(t) \), or \( r(t) = -s(t) + z(t) \), where \( z(t) \) is white Gaussian noise with \( \mathbb{E}[z(t)] = 0 \) and \( R_z(\tau) = \mathbb{E}[z(t)z^*(\tau)] = 2N_0\delta(t-\tau) \). The optimum receiver computes the correlation metric

\[ U = \text{Re} \left[ \int_0^T r(t)s^*(t)\, dt \right] \]

and compares \( U \) with a threshold \( A \) and a threshold \(-A\). If \( U > A \), the decision is made that \( s(t) \) was sent. If \( U < -A \), the decision is made in favor of \(-s(t)\). If \(-A < U < A \), the decision is made in favor of 0.

1. Determine the three conditional probabilities of error: \( P_e \) given that \( s(t) \) was sent, \( P_e \) given that \(-s(t)\) was sent, and \( P_e \) given that 0 was sent.
2. Determine the average probability of error \( P_e \) as a function of the threshold \( A \), assuming that the three symbols are equally probable a priori.
3. Determine the value of \( A \) that minimizes \( P_e \).

3. (Problems 4.24 in the textbook)

Three equiprobable messages \( m_1, m_2, \text{ and } m_3 \) are to be transmitted over an AWGN channel with noise power spectral density \( \frac{1}{2}N_0 \). The messages are

\[ s_1(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \quad s_2(t) = -s_3(t) = \begin{cases} 1 & 0 \leq t \leq \frac{1}{2}T \\ -1 & \frac{1}{2}T < t \leq T \\ 0 & \text{otherwise} \end{cases} \]

1. What is the dimensionality of the signal space?
2. Find an appropriate basis for the signal space.
3. Draw the signal constellation for this problem.
4. Derive and sketch the optimal decision regions \( R_1, R_2, \text{ and } R_3 \).
5. Which of the three messages is most vulnerable to errors and why? In other words, which of \( P(\text{error} | m_i \text{ transmitted}), i = 1, 2, 3, \) is largest?
The discrete sequence
\[ r_k = \sqrt{E} c_k + n_k, \quad k = 1, 2, \ldots, n \]
represents the output sequence of samples from a demodulator, where \( c_k = \pm 1 \) are elements of one of two possible code words, \( c_1 = [1 \ 1 \ \cdots \ 1] \) and \( c_2 = [1 \ 1 \ \cdots \ 1 \ -1 \ \cdots \ -1] \). The code word \( c_2 \) has \( w \) elements that are +1 and \( n - w \) elements that are −1, where \( w \) is some positive integer. The noise sequence \( \{ n_k \} \) is white Gaussian with variance \( \sigma^2 \).

1. What is the optimum maximum-likelihood detector for the two possible transmitted signals?

2. Determine the probability of error as a function of the parameters (\( \sigma^2, E, w \)).

3. What is the value of \( w \) that minimizes the error?