Solutions of Sample Problems (24th Nov.)

1. Problem 4.36

1. The noncoherent envelope detector for the on-off keying signal is depicted in the next figure.

2. If \( s_0(t) \) is sent, then the received signal is \( r(t) = n(t) \) and therefore the sampled outputs \( r_c, r_s \) are zero-mean independent Gaussian random variables with variance \( \frac{N_0}{2} \). Hence, the random variable \( r = \sqrt{r_c^2 + r_s^2} \) is Rayleigh distributed and the PDF is given by:

\[
p(r|s_0(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} = \frac{2r}{N_0} e^{-\frac{r^2}{2N_0}}
\]

If \( s_1(t) \) is transmitted, then the received signal is:

\[
r(t) = \sqrt{\frac{2\xi_b}{T_b}} \cos(2\pi f_d t + \phi) + n(t)
\]

Crosscorrelating \( r(t) \) by \( \sqrt{\frac{2}{T}} \cos(2\pi f_d t) \) and sampling the output at \( t = T \), results in:

\[
r_c = \int_0^T r(t) \sqrt{\frac{2}{T}} \cos(2\pi f_d t) dt
\]

\[
= \int_0^T 2\sqrt{\frac{2\xi_b}{T_b}} \cos(2\pi f_d t + \phi) \cos(2\pi f_d t) dt + \int_0^T n(t) \sqrt{\frac{2}{T}} \cos(2\pi f_d t) dt
\]

\[
= \frac{2\sqrt{\frac{2\xi_b}{T_b}}}{T_b} \int_0^T \frac{1}{2} (\cos(2\pi f_d t + \phi) + \cos(\phi)) dt + n_c
\]

\[
= \sqrt{\xi_b} \cos(\phi) + n_c
\]

where \( n_c \) is zero-mean Gaussian random variable with variance \( \frac{N_0}{4} \). Similarly, for the quadrature component we have:

\[r_s = \sqrt{\xi_b} \sin(\phi) + n_s\]

The PDF of the random variable \( r = \sqrt{r_c^2 + r_s^2} = \sqrt{\xi_b + n_c^2 + n_s^2} \) follows the Rician distribution:

\[
p(r|s_1(t)) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} I_0 \left( \frac{r}{\sigma^2} \right) = \frac{2r}{N_0} e^{-\frac{r^2}{2N_0}} I_0 \left( \frac{2r}{\sqrt{\xi_b}} \right)
\]

3. For equiprobable signals the probability of error is given by:

\[
P(\text{error}) = \frac{1}{2} \int_{-\infty}^{N_0^r} p(r|s_1(t)) dr + \frac{1}{2} \int_{N_0^r}^{\infty} p(r|s_0(t)) dr
\]
Since \( r > 0 \) the expression for the probability of error takes the form

\[
P(\text{error}) = \frac{1}{2} \int_0^{\sqrt{V_T}} pr(s_1(t)) dr + \frac{1}{2} \int_{\sqrt{V_T}}^\infty p(r|s_0(t)) dr
\]

\[
= \frac{1}{2} \int_0^{\sqrt{V_T}} \frac{r}{\sigma^2} e^{-\frac{r^2 + s_0^2}{2\sigma^2}} I_0 \left( \frac{r \sqrt{s_0^2}}{\sigma^2} \right) dr + \frac{1}{2} \int_{\sqrt{V_T}}^\infty \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr
\]

The optimum threshold level is the value of \( V_T \) that minimizes the probability of error. However, when \( \frac{\hat{p}}{N_0} \gg 1 \) the optimum value is close to \( \frac{\sqrt{2}}{2} \) and we will use this threshold to simplify the analysis. The integral involving the Bessel function cannot be evaluated in closed form. Instead of \( I_0(x) \) we will use the approximation:

\[
I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}
\]

which is valid for large \( x \), that is for high SNR. In this case:

\[
\frac{1}{2} \int_0^{\sqrt{V_T}} \frac{r}{\sigma^2} e^{-\frac{r^2 + s_0^2}{2\sigma^2}} I_0 \left( \frac{r \sqrt{s_0^2}}{\sigma^2} \right) dr \approx \frac{1}{2} \int_0^{\sqrt{V_T}} \frac{\sqrt{V_T}}{2\sigma \sqrt{s_0}} e^{-\frac{(r \sqrt{s_0^2})^2}{2\sigma^2}} dr
\]

This integral is further simplified if we observe that for high SNR, the integrand is dominant in the vicinity of \( \sqrt{s_0} \) and therefore, the lower limit can be substituted by \( -\infty \). Also

\[
\sqrt{\frac{r}{2\pi \sigma^2 \sqrt{s_0}}} \approx \sqrt{\frac{1}{2\pi \sigma^2}}
\]

and therefore:

\[
\frac{1}{2} \int_0^{\sqrt{V_T}} \frac{\sqrt{V_T}}{2\pi \sigma \sqrt{s_0}} e^{-\frac{(r \sqrt{s_0^2})^2}{2\sigma^2}} dr \approx \frac{1}{2} \int_{-\infty}^{\sqrt{V_T}} \frac{1}{2\pi \sigma^2} e^{-\frac{(r \sqrt{s_0^2})^2}{2\sigma^2}} dr
\]

\[
= \frac{1}{2} Q \left( \sqrt{\frac{\sqrt{s_0}}{2N_0}} \right)
\]

Finally:

\[
P(\text{error}) = \frac{1}{2} Q \left[ \sqrt{\frac{\sqrt{s_0}}{2N_0}} \right] + \frac{1}{2} \int_{\sqrt{\sqrt{s_0}}}^\infty 2r e^{-\frac{r^2}{2N_0}} dr
\]

\[
\leq \frac{1}{2} Q \left[ \sqrt{\frac{\sqrt{s_0}}{2N_0}} \right] + \frac{1}{2} e^{-\frac{\sqrt{s_0}}{2N_0}}
\]
2.

Problem 4.50

1. Since $a$ takes only nonnegative values the threshold of the binary antipodal signaling scheme always remains at zero and the decision rule is similar to an AWGN channel.

2. For a given value of $a$ we have $P_b = Q\left(a\sqrt{E_b/N_0}\right)$. We have to average the above value over all $a$'s to obtain the error probability. This is done similar to the integration in part 1 of Problem 4.44 to give

$$P_b = \frac{1}{2} \left(1 - \sqrt{\frac{E_b/N_0}{1 + E_b/N_0}}\right)$$

where we have used the fact that $\sigma^2$ parameter in the Rayleigh distribution given above is $1/2$.

3. For large SNR values

$$\frac{E_b/N_0}{1 + E_b/N_0} = \frac{1}{1 + 1/(E_b/N_0)} \approx 1 - 1/(E_b/N_0) \approx (1 - 1/(2 E_b/N_0))^2$$

Hence,

$$P_b \approx \frac{1}{2} (1 - 1 + 1/(2 E_b/N_0)) = \frac{1}{4 E_b/N_0}$$

4. For an AWGN channel an error probability of $10^{-5}$ is achieved at $E_b/N_0$ of 9.6 dB. For a fading channel from $10^{-5} = \frac{1}{4E_b/N_0}$ we have $E_b/N_0 = 10^5/4 = 25000$ or 44 dB, a difference of 34.4 dB.

5. For orthogonal signaling and noncoherent detection $P_b = \frac{1}{2} e^{-a^2 E_b/2 N_0}$. We need to average this using the PDF of $a$ to determine the error probability. This is done similar to part 5 of Problem 4.44. The result is $P_b = 1/(2 + E_b/N_0)$.

3.
4.

**Problem 5.8**

An on-off keying signal is represented as:

\[ s_1(t) = A \cos(2\pi f_c t + \phi_c), \quad 0 \leq t \leq T \text{ (binary 1)} \]
\[ s_2(t) = 0, \quad 0 \leq t \leq T \text{ (binary 0)} \]

Let \( r(t) \) be the received signal, that is \( r(t) = s(t; \phi_c) + n(t) \) where \( s(t; \phi_c) \) is either \( s_1(t) \) or \( s_2(t) \) and \( n(t) \) is white Gaussian noise with variance \( \frac{N_0}{2} \). The likelihood function, that is to be maximized with respect to \( \phi_c \) over the interval \([0, T]\), is proportional to:

\[ L(\phi_c) = \exp \left[ -\frac{2}{N_0} \int_0^T [r(t) - s(t; \phi_c)]^2 dt \right] \]

Maximization of \( L(\phi_c) \) is equivalent to the maximization of the log-likelihood function:

\[ L_L(\phi_c) = -\frac{2}{N_0} \int_0^T [r(t) - s(t; \phi_c)]^2 dt \]
\[ = -\frac{2}{N_0} \int_0^T r^2(t) dt + \frac{4}{N_0} \int_0^T r(t) s(t; \phi_c) dt - \frac{2}{N_0} \int_0^T s^2(t; \phi_c) dt \]

Since the first term does not involve the parameter of interest \( \phi_c \) and the last term is simply a constant equal to the signal energy of the signal over \([0, T]\) which is independent of the carrier phase, we can carry the maximization over the function:

\[ V(\phi_c) = \int_0^T r(t) s(t; \phi_c) dt \]

Note that \( s(t; \phi_c) \) can take two different values, \( s_1(t) \) and \( s_2(t) \), depending on the transmission of a binary 1 or 0. Thus, a more appropriate function to maximize is the average log-likelihood

\[ \overline{V}(\phi_c) = \frac{1}{2} \int_0^T r(t) s_1(t) dt + \frac{1}{2} \int_0^T r(t) s_2(t) dt \]

Since \( s_2(t) = 0 \), the function \( \overline{V}(\phi_c) \) takes the form:

\[ \overline{V}(\phi_c) = \frac{1}{2} \int_0^T r(t) A \cos(2\pi f_c t + \phi_c) dt \]

Setting the derivative of \( \overline{V}(\phi_c) \) with respect to \( \phi_c \) equal to zero, we obtain:

\[ \frac{\partial \overline{V}(\phi_c)}{\partial \phi_c} = 0 = \frac{1}{2} \int_0^T r(t) A \sin(2\pi f_c t + \phi_c) dt \]
\[ = \cos \phi_c \frac{1}{2} \int_0^T r(t) A \sin(2\pi f_c t) dt + \sin \phi_c \frac{1}{2} \int_0^T r(t) A \cos(2\pi f_c t) dt \]

Thus, the maximum likelihood estimate of the carrier phase is:

\[ \hat{\phi}_{c, ML} = - \arctan \left( \frac{\int_0^T r(t) \sin(2\pi f_c t) dt}{\int_0^T r(t) \cos(2\pi f_c t) dt} \right) \]