Solutions to sample example (Oct 27)

3.19

1. Since $\mu_a = 0, \sigma_a^2 = 1$, we have: $S_{aa}(f) = \frac{1}{T} |G(f)|^2$. But:

$$
G(f) = \frac{T}{2} \frac{\sin \pi f T/2}{\pi f T/2} e^{-j2\pi f T/4} - \frac{T}{2} \frac{\sin \pi f T/2}{\pi f T/2} e^{-j2\pi f 3T/4}
$$

$$
= \frac{T}{2} \frac{\sin \pi f T/2}{\pi f T/2} e^{-j\pi f T} (2j \sin \pi f T/2)
$$

$$
= jT \frac{\sin^2 \pi f T/2}{\pi f T/2} e^{-j\pi f T} \Rightarrow
$$

$$
|G(f)|^2 = T^2 \left( \frac{\sin^2 \pi f T/2}{\pi f T/2} \right)^2 \Rightarrow
$$

$$
S_{aa}(f) = T \left( \frac{\sin^2 \pi f T/2}{\pi f T/2} \right)^2
$$

2. For non-independent information sequence the power spectrum of $s(t)$ is given by: $S_{ss}(f) = \frac{1}{T} |G(f)|^2 S_{bb}(f)$. But:

$$
R_{bb}(m) = E[b_{n+m}b_n] = E[a_{n+m}a_n] + kE[a_{n+m-1}a_n] + k^2 E[a_{n+m-1}a_{n-1}]
$$

$$
= \left\{
\begin{array}{l}
1 + k^2, \quad m = 0 \\
1 + k, \quad m = \pm 1 \\
0, \quad \text{o.w.}
\end{array}
\right.
$$

Hence:

$$
S_{bb}(f) = \sum_{m=-\infty}^{\infty} R_{bb}(m) e^{-j2\pi f m T} = 1 + k^2 + 2k \cos 2\pi f T
$$

We want:

$$
S_{as}(1/T) = 0 \Rightarrow S_{bb}(1/T) = 0 \Rightarrow 1 + k^2 + 2k = 0 \Rightarrow k = -1
$$

and the resulting power spectrum is:

$$
S_{aa}(f) = 4T \left( \frac{\sin^2 \pi f T/2}{\pi f T/2} \right)^2 \sin^2 \pi f T
$$

3. The requirement for zeros at $f = l/4T, l = \pm 1, \pm 2, \ldots$ means: $S_{bb}(l/4T) = 0 \Rightarrow 1 + k^2 + 2k \cos \pi l/2 = 0$, which cannot be satisfied for all $l$. We can avoid that by using precoding in the
form: $b_n = a_n + ka_{n-4}$. Then:

$$R_{b_0}(m) = \begin{cases} 1 + k^2, & m = 0 \\ k, & m = \pm 4 \\ 0, & \text{o.w.} \end{cases} \Rightarrow S_{b_0}(f) = 1 + k^2 + 2k \cos 2\pi f 4T$$

and, similarly to (b), a value of $k = -1$, will zero this spectrum in all multiples of $1/4T$.

3.25

1. We have

$$R_a(m) = E[a_{n+m}a_n]$$

$$= \begin{cases} E[a_n^2] & m = 0 \\ (E[a_n])^2 & m \neq 0 \end{cases}$$

$$= \begin{cases} \frac{5}{4} & m = 0 \\ \frac{1}{16} & m \neq 0 \end{cases}$$

Using $S_a(f) = \sum_{m=-\infty}^{\infty} R_a(m)e^{-j2\pi fmT}$, we have

$$S_a(f) = \frac{19}{16} + \frac{1}{16} \sum_{m=-\infty}^{\infty} e^{-j2\pi fmT}$$

$$= 1916 + 116Tm = -\infty \ldots \infty \delta(f - mT)$$

and since $g(t) = \text{sinc}(t/T)$, we have $G(f) = T\Pi(Tf)$, hence $|G(f)|^2 = T^2 \Pi(Tf)$ and

$$S_{v_1}T2\pi Tf \quad [1916 + 116Tm = -\infty \ldots \infty \delta(f - mT)]$$

resulting in

$$S_a(f) = \frac{19}{16} T\Pi(Tf) + \frac{1}{16} \delta(f)$$

2. The power spectral density is multiplied by $|1 + e^{-j2\pi fT} - e^{-j2\pi fT}|^2 = 3 - 2 \cos(4\pi fT)$. Therefore

$$S_a(f) = \frac{19}{16} (3 - 2 \cos 4\pi fT) T\Pi(f) + \frac{1}{16} \delta(f)$$

3. In this case $S_a(f)$ is multiplied by $|1 + j e^{-j2\pi fT}|^2 = (2 + 2 \sin 2\pi fT)$ and

$$S_a(f) = \frac{19}{8} (1 + \sin 2\pi fT) T\Pi(f) + \frac{1}{8} \delta(f)$$

4.5
1. Note that \( s_2(t) = 2s_1(t) \) and \( s_3(t) = 0 \), hence the system is PAM and a singular basis function of the form \( \phi_1(t) = \frac{1}{A\sqrt{T}} s_1(t) \) would work

\[
\phi(t) = \begin{cases} 
\frac{1}{\sqrt{T}} & 0 < t \leq T/3 \\
-\frac{1}{\sqrt{T}} & T/3 \leq t < T 
\end{cases}
\]

Assuming \( E_1 = A^2T \), we have \( s_3 = 0 \), \( s_1 = \sqrt{E_1} \), \( s_2 = 2\sqrt{E_1} \). The constellation is shown below.

2. For equiprobable messages the optimal decision rule is the nearest neighbor rule and the perpendicular bisectors are the boundaries of the decision regions as indicated in the figure.

3. This is ternary PAM system with the distance between adjacent points in the constellation being \( d = \sqrt{E_1} = A\sqrt{T} \). The average energy is \( E_{\text{avg}} = \frac{1}{3}(0 + A^2T + 4A^2T) = \frac{5}{3}A^2T \), and \( E_{\text{bavg}} = E_{\text{avg}}/\log_2 3 = \frac{5}{3\log_2 3}A^2T \), from which we obtain

\[
d^2 = \frac{3\log_2 3}{5} E_{\text{bavg}} \approx 0.951 E_{\text{bavg}}
\]

The error probability of the optimal detector is the average of the error probabilities of the three signals. For the two outer signals error probability is \( P(n > d/2) = Q\left(\frac{d/2}{\sqrt{N_0/2}}\right) \) and for the middle point \( s_1 \) it is \( P(|n| > d/2) = 2Q\left(\frac{d/2}{\sqrt{N_0/2}}\right) \). From this,

\[
P_s = \frac{4}{3} Q\left(\sqrt{\frac{d^2}{2N_0}}\right) = \frac{4}{3} Q\left(\sqrt{\frac{0.951 E_{\text{bavg}}}{2N_0}}\right) = 43\sqrt{\frac{0.475 E_{\text{bavg}}}{2N_0}}
\]

4. \( R = R_s \log_2 M = 3000 \times \log_2 3 \approx 4755 \) bps.