1. Give a BPSK-modulated communication system below, where \( T \) is the width of a channel symbol. Assume that \( \int_0^T \cos(4\pi f_c t) dt = 0 \).

(a) (6 pt) Show that the transmission power of the BPSK modulated (passband) signal is equal to \( \mathcal{E} \).

(Hint: The signal power of a transmitted signal \( s(t) \) is defined as \( \int_0^T s^2(t) dt \). From this case, you should learn that the SNR definition commonly used in the literature is based on a passband signal with power \( \mathcal{E} \), which is transmitted over the AWGN noise with \( N_0/2 \) power spectrum density.)

(b) (6 pt) As shown in the above figure, the one dimensional received vector \( r \) is equal to the (one-dimensional) signal vector \( s \) plus the noise sample. Suppose that the noise process \( n(t) \) is a zero-mean white Gaussian process with power spectrum density \( N_0/2 \). Prove that \( s = \pm \sqrt{\mathcal{E}} \) and \( n \) is Gaussian distributed with mean zero and variance \( N_0/2 \).

(c) (6 pt) Assume a uniform prior. Show that the minimum error probability attainable for the above receiver is \( \Phi(-\sqrt{2\gamma}) \), where \( \gamma = \mathcal{E}/N_0 \) and \( \Phi(.) \) represents the cdf of the standard normal distribution.

(Now you should know why we can directly write the input of the decision maker as \( r = \pm \sqrt{\mathcal{E}} + n \).)

Answer:

(a)

\[
\int_0^T \left[ \pm \sqrt{\frac{2\mathcal{E}}{T}} \cos(2\pi f_c t) \right]^2 dt = \int_0^T \frac{2\mathcal{E}}{T} \cos^2(2\pi f_c t) dt
\]

\[
= \int_0^T \frac{2\mathcal{E}}{T} \left( \frac{1 + \cos(4\pi f_c t)}{2} \right) dt
\]

\[
= \int_0^T \frac{\mathcal{E}}{T} dt + \frac{\mathcal{E}}{T} \int_0^T \cos(4\pi f_c t) dt
\]

\[
= \mathcal{E}
\]
By definition, \( n = \int_0^T n(t) \sqrt{\frac{2}{T}} \cos(2\pi f c t) dt \) is Gaussian distributed. The proof of its mean being zero is easy; hence, we omit it. Its variance can be calculated below.

\[
E[n^2] = E \left[ \int_0^T \int_0^T n(t) n(s) \sqrt{\frac{2}{T}} \cos(2\pi f c t) \cos(2\pi f c s) dt ds \right] \\
= \frac{2}{T} \int_0^T \int_0^T E[n(t)n(s)] \cos(2\pi f c t) \cos(2\pi f c s) dt ds \\
= \frac{2}{T} \int_0^T \int_0^T N_0 \delta(t-s) \cos(2\pi f c t) \cos(2\pi f c s) dt ds \\
= \frac{N_0}{T} \cos^2(2\pi f c t) dt \\
= \frac{N_0}{2} 
\]

(c) See slide chap5a-5.

2. Suppose the signal in problem 1 is transmitted with propagation delay \( \tau \in (0,T) \). Assume that 
\[
\int_0^T \cos(4\pi f c t) dt = \int_0^T \sin(4\pi f c t) dt = 0 .
\]

(a) (6 pt) Derive \( s \) and \( n \) under the fixed propagation delay \( \tau \). (Note that in your derivation, please denote \( \phi = -2\pi f c \tau \).)

(b) (6 pt) Assume a uniform prior. Derive the minimum error probability under the propagation delay \( \tau \in (0,T) \).

(c) (6 pt) Under the propagation delay \( \tau \in (0,T) \), determine the “enlarge-ratio” of the signal power to obtain the same error probability as problem 1(c). (“Enlarge-ratio” = new power in 2(c) / \( E \) in problem 1.)

**Answer:**

(a) 
\[
s(t) = \pm \sqrt{\frac{2E}{T}} g(t) \cos(2\pi f c t) \Rightarrow s(t-\tau) = \pm \sqrt{\frac{2E}{T}} g(t-\tau) \cos(2\pi f c t + \phi)
\]

Hence,
The derivation of \( n \) is the same as that in problem 1.

(b) The error probability is
\[
\Phi \left( \left[ \frac{T - \tau}{T} \right] \cdot \left| \cos(\phi) \right| \cdot \sqrt{2} \gamma \right),
\]
where \( \gamma = \mathcal{E} / N_0 \) and \( \Phi(\cdot) \) represents the cdf of the standard normal distribution.

(c) Let \( \mathcal{E}' \) be the new signal power. Then apparently, \( \mathcal{E}'/\mathcal{E} = \frac{T}{(T - \tau) \cos(\phi)} \).

3. (6 pt) Characterize the possible behavior of optimal error probability for \( M \)-ary orthogonal signal, as \( M \) tends to infinity, if the signal to noise ratio per information bit is smaller than the Shannon limit –1.6dB.
Answer: If the signal to noise ratio per information bit is smaller than the Shannon limit –1.6dB, the optimal error probability for \( M \)-ary orthogonal signal, as \( M \) tends to infinity, is bounded away from zero.
In other words, there exists an universal (in \( M \)) non-negative lower bound for the error probability of \( M \)-ary orthogonal signal.

4. (16 pt) Suppose the received lowpass equivalent signal, due to propagation delay, is of the form
\[
r_j(t) = e^{-j\phi} \cdot I \cdot g(t - \tau) + z(t) = s_j(t; \phi, \tau) + z(t),
\]
where \( I \) is either +1 or –1, the distribution of \( r_j(t) \) given \( \phi \) and \( \tau \) is
\[
p(r_j(t) | \phi, \tau) = \left( \frac{1}{\sqrt{2\pi N_0}} \right)^{2N} \exp \left\{ -\frac{1}{2} \left[ (r_j(t) - s_j(t; \phi, \tau))^2 / N_0 \right] \right\},
\]
and the observation interval \( T_0 = T \). Suppose that \( g(t) \) equals one for \( 0 \leq t < T \), and zero, otherwise. Find the decision directed maximum-likelihood estimator of \( \phi \) and \( \tau \) based on a specific received value \( r_j(t) = 1 + j \).
Answer:
\[
(\hat{\phi}_{ML}, \hat{\tau}_{ML}) = \arg \min_{\phi, \tau} \int_0^T \left[ r_j(t) - s_j(t; \phi, \tau) \right] \left[ r_j(t) - s_j(t; \phi, \tau) \right]^* dt / N_0,
\]
\[
= \arg \min_{\phi, \tau} \int_0^T \left\{ r_j(t)^2 + I^2 g(t - \tau) - r_j^* (t) e^{-j\phi} g(t - \tau) - r_j(t) e^{j\phi} g(t - \tau) \right\} dt
\]
\[
= \arg \min_{\phi, \tau} \int_0^T \left( \left( r_j(t)^2 + g(t - \tau) - 2I \cdot g(t - \tau) \Re \{ r_j^* (t) e^{-j\phi} \} \right)^2 \right) dt
\]
\[
= \arg \min_{\phi, \tau} \int_0^T \left( \left( r_j(t)^2 + g(t - \tau) - 2I \cdot g(t - \tau) \Re \{ (1 - j)e^{-j\phi} \} \right)^2 \right) dt
\]
\[
= \arg \min_{\phi, \tau} \left[ (1 - 2I[\cos(\phi) - \sin(\phi)])^T g(t - \tau) dt \right)
\]
\[
= \arg \min_{\phi, \tau} \left[ (1 - 2I[\cos(\phi) - \sin(\phi)]) (T - |\tau|) \cdot 1(-T < \tau < T),
\]
where \( \mathbf{1}(\cdot) \) is the indicator function. By taking the derivative of the above minimizing quantity with respect to \( \hat{\phi}_M \), it is either \( 3\pi/4 \) or \( 7\pi/4 \). Substitute these two values into the above minimizing quantity, we obtain that this quantity is either 
\[
(\mathbf{1}(-T < \tau < T) \cdot (1 - 2\sqrt{2}I)(T - |\tau|)) \cdot \mathbf{1}(-T < \tau < T) \cdot 1
\]
or 
\[
(\mathbf{1}(-T < \tau < T) \cdot (1 + 2\sqrt{2}I)(T - |\tau|)) \cdot \mathbf{1}(-T < \tau < T) \cdot 1
\]. Hence,
\[
\hat{\phi}_M = \begin{cases} 
3\pi / 4, & I = +1; \\
7\pi / 4, & I = -1.
\end{cases}
\]
Also, no matter what \( I \) is, \( \hat{\tau}_M = 0. \) (From this result, you learn that \( \hat{\phi}_M \) does not necessarily equal \( -2\pi\hat{\tau}_M \); even if both are ML estimates.)

5. (12 pt) Suppose that the noise-free relation between the (lowpass equivalent) channel output \( r_i(t) \) and the (lowpass equivalent) channel input \( s_i(t) \) is
\[
r_i(t) = \int_{-\infty}^{\infty} \alpha(\tau; t)s_i(t - \tau) e^{-j2\pi\tau t} d\tau.
\]
Also, we know that the channel output and the channel input can be characterized by “convolution” operation as:
\[
r_i(t) = \int_{-\infty}^{\infty} c_j(\tau; t) s_i(t - \tau) d\tau
\]
where \( c_j(\tau; t) \) is the channel impulse response. By comparing these two formulas, express or formulate the (lowpass equivalent) channel impulse response \( c_j(\tau; t) \) in terms of \( \alpha(\tau; t) \).

Answer:
Since \( r_i(t) = \int_{-\infty}^{\infty} \alpha(\tau; t)s_i(t - \tau) e^{-j2\pi\tau t} d\tau = \int_{-\infty}^{\infty} \alpha(\tau; t)e^{-j2\pi\tau t} j_i(t - \tau) d\tau \),

Therefore, \( c_j(\tau; t) = \alpha(\tau; t)e^{-j2\pi\tau t} \).

6. (12 pt) Suppose that for a direct sequence spread spectrum system, the transmitted codeword is \( [c_0, c_1, c_2] \), and the PN sequence is \( [b_0, b_1, b_2] \), where \( a_i \) and \( b_i \) are elements of \( \{0,1\} \). After performing the chip-wise XOR operation between \( [c_0, c_1, c_2] \) and \( [b_0, b_1, b_2] \), i.e., \( a_i = c_i \oplus b_i \) for \( 0 \leq i \leq 2 \), the communication system antipodally sends out the resultant chip sequence with pulse shape \( g(t) \), where \( g(t) = 1 \) for \( 0 \leq t < 2T_c \), and zero, otherwise (Note that the multiplicative constant 2 is not mis-printed; there is indeed a 2 before \( T_c \)). Also, \( T_c \) is the chip duration. Prove that the transmitted signal can be represented as
\[
s_i(t) = \sum_{i=0}^{2} (c_i - 1) w(t - iT_c) \quad \text{and} \quad w(t) = 1 \quad \text{for} \quad 0 \leq t < T_c.
\]

Answer: The transmitted symbol corresponding to \( i \)th interval is
\[
s_i(t) = (1 - 2a_i) g(t - iT_c)
\]
\[
= [1 - 2(b_i \oplus c_i)] g(t - iT_c)
\]
\[
= (1 - 2b_j)(1 - 2c_j) g(t - iT_c)
\]
\[
= (2b_j - 1)(2c_j - 1) g(t - iT_c)
\]
\[
= [(2b_j - 1) g(t - iT_c)] \times [(2c_j - 1) g(t - iT_c)]
\]
\[
= [(2b_j - 1) w(t - iT_c) + (2b_j - 1) w(t - (i + 1)T_c)] \times [(2c_j - 1) w(t - iT_c) + (2c_j - 1) w(t - (i + 1)T_c)]
\]
\[
= (2b_j - 1) w(t - iT_c)(2c_j - 1) w(t - iT_c) + (2b_j - 1) w(t - (i + 1)T_c)(2c_j - 1) w(t - iT_c)
\]
\[
+ (2b_j - 1) w(t - iT_c)(2c_j - 1) w(t - (i + 1)T_c) + (2b_j - 1) w(t - (i + 1)T_c)(2c_j - 1) w(t - iT_c)
\]
\[
= (2b_j - 1) w(t - iT_c)(2c_j - 1) w(t - iT_c) + (2b_j - 1) w(t - (i + 1)T_c)(2c_j - 1) w(t - (i + 1)T_c)
\]
\[
= p_i(t) c_i(t) + p_i(t - T_c) c_i(t - T_c)
\]
where \( p_i(t) = (2b_j - 1) w(t - iT_c) \) and \( c_i(t) = (2c_j - 1) w(t - iT_c) \). Hence,
7. (6 pt) Under the worst-case pulsed jammer, sort the following decoding/decision scheme in the order of descending cutoff rate:

1. Soft-decision with complete knowledge of jammer state information;
2. Hard-decision with complete knowledge of jammer state information;
3. Soft-decision without knowledge of jammer state information;
4. Hard-decision without knowledge of jammer state information.

Answer: (1) > (2) > (4) > (3).

8. (a) (6 pt) Suppose that the (lowpass equivalent) channel impulse response is \( c_i(\tau; t) = \alpha(\tau; t)e^{-j2\pi f_c \tau} \).

Assume that \( \alpha(\tau; t) \) is zero-mean, and WSS in \( t \). Let

\[
\phi_{\alpha}(\tau, \Delta t) = \frac{1}{2} E[\alpha^*(\tau; t)\alpha(\tau; t + \Delta t)] = \begin{cases} \frac{10^7}{10^{-7}} (10^{-7} - \tau) & \text{for } 0 \leq \tau \leq 100ns; \\ 0, & \text{otherwise} \end{cases}
\]

be the autocorrelation function of \( \alpha(\tau; t) \). Also, assume that \( \alpha(\tau; t_1) \) and \( \alpha(\tau; t_2) \) are uncorrelated whenever \( \tau \neq \overline{\tau} \). Find the autocorrelation function of the (complex-valued) \( c_i(\tau; t) \).

(Hint: \( \phi_{c_i}(\tau; \overline{\tau}; \Delta t) = \frac{1}{2} E[c_i^*(\tau; t)c_i(\overline{\tau}; t + \Delta t)] \))

(b) (6 pt) What is the delay spread of this (uncorrelated-scattering) multipath fading channel.

Answer:

\[
\phi_{c_i}(\tau; \overline{\tau}; \Delta t) = \frac{1}{2} E[c_i^*(\tau; t)c_i(\overline{\tau}; t + \Delta t)] = \frac{1}{2} E[\alpha^*(\tau; t)e^{-j2\pi f_c \tau} \times \alpha(\overline{\tau}; t + \Delta t)e^{-j2\pi f_c \tau}] = \frac{1}{2} E[\alpha^*(\tau; t)\alpha(\tau; t + \Delta t)]e^{-j2\pi f_c (\tau - \overline{\tau})} = \frac{1}{2} E[\alpha^*(\tau; t)\alpha(\tau; t + \Delta t)]\delta(\tau - \overline{\tau}) = \phi_{\alpha}(\tau; \Delta t) \times \delta(\tau - \overline{\tau}).
\]

When \( \Delta t = 0 \), \( \phi_{c_i}(\tau; \overline{\tau}; \Delta t = 0) = \phi_{\alpha}(\tau; \Delta t = 0) \times \delta(\tau - \overline{\tau}) \). Hence, the delay spread of this channel is 100ns.