2006 Spring Midterm for Advanced Probability for Communications

Each problem costs you 10 points.

1. (Section 6)

(a) It is straightforward that if \( \{A_i\}_{i=1}^n \) are disjoint with probability one, then

\[
P \left( \bigcup_{i=1}^n A_i \right) = \sum_{i=1}^n P(A_i). \tag{1}\]

Prove that the converse statement is also true, i.e., (1) implies that \( \{A_i\}_{i=1}^n \) are disjoint with probability one.

(Hint: List all the subset \( I_1, I_2, \ldots, I_{2^n-1} \) of \( N \triangleq \{1, 2, 3, \ldots, n\} \) except the empty set. Define \( B_j \triangleq \bigcap_{i \in I_j} A_i - \bigcap_{i \in N-I_j} A_i \). Then, find the general representations of \( P \left( \bigcup_{i=1}^n A_i \right) \) and \( \sum_{i=1}^n P(A_i) \) in terms of \( P(B_j) \), respectively.)

(b) If \( \{A_i\}_{i=1}^\infty \) are disjoint, then what can we say about \( P(\limsup_{i \to \infty} A_i) \) from the two Borel-Cantelli lemmas?

(Hint: Problem 1(a) is valid even if \( n \) is replaced by \( \infty \).)

(c) Define \( X_i(\omega) = 1 \) if \( \omega \in A_i \) and zero, otherwise. Does the strong law (i.e., \( (1/n) \sum_{i=1}^n X_i \) converges with probability one to some constant) hold for \( \{X_i\}_{i=1}^\infty \)?

(Hint: The strong law requires the validity of the first Borel-Cantelli lemma.)

Proof:

(a) That \( \{B_j\}_{j=1}^{2^n-1} \) are disjoint and \( \bigcup_{i=1}^n A_i = \bigcup_{j=1}^{2^n-1} B_j \) jointly imply that

\[
P \left( \bigcup_{i=1}^n A_i \right) = \sum_{j=1}^{2^n-1} P(B_j).\]

Also, \( A_i = \bigcup_{j: i \in I_j} B_j \) for \( 1 \leq i \leq n \) implies

\[
\sum_{i=1}^n P(A_i) = \sum_{i=1}^n \sum_{\{j: i \in I_j\}} P(B_j) = \sum_{j=1}^{2^n-1} |I_j| \cdot P(B_j).
\]
The validity of (1) then gives
\[ \sum_{j=1}^{2^n-1} (|I_j| - 1) \cdot P(B_j) = 0. \]

The proof is completed by noting that the sum of nonnegative \((|I_j| - 1) \cdot P(B_j)\) equal zero implies all terms should equal zero, and hence, \(P(B_j)\) is positive only when \(|I_j| = 1\).

(b) Since \(\{A_i\}_{i=1}^{\infty}\) are disjoint,
\[ \sum_{i=1}^{\infty} P(A_i) = P\left( \bigcup_{i=1}^{\infty} A_i \right) < 1. \]

Then, by the first Borel-Cantelli lemma, \(P(\limsup_{i \to \infty} A_i) = 0\).

(c) \(P(\limsup_{i \to \infty} A_i) = 0\) implies
\[ P(\{\omega : X_i(\omega) = 1 \text{ for finitely many } i\}) = 1. \]

Therefore,
\[ P\left( \left\{ \omega : \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i(\omega) = 0 \right\} \right) = 1, \]

or equivalently,
\[ \Pr\left( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i = 0 \right) = 1; \]

hence, the strong law holds.

2. (Section 9)

(a) Fix a random variable \(Y\) with probability density function \(f_Y(y)\).
Prove that for any strictly increasing positive function \(g(\cdot)\) with \(g(0) = 1\), and for any positive integer \(k\),
\[ E[g^k(Y)] \geq \Pr[Y \geq 0] = E[g^k(Y)] \int_0^{\infty} \frac{f_W(y)}{g^k(y)} \, dy. \]
where
\[
f_W(y) \triangleq \int_{-\infty}^{\infty} \frac{g^k(y)f_Y(y)}{g^k(y)f_Y(y)dy} = \frac{g^k(y)f_Y(y)}{E[g^k(Y)]}.
\]

(Hint: Upper bound can be proved by Markov inequality. Lower bound can be proved by the definition of \(f_W(y)\).)

(b) Prove that
\[0 \leq \int_0^\infty \frac{f_W(y)}{g^k(y)}dy \leq 1.\]

Proof:

(a) By Markov inequality,
\[
\Pr[Y \geq 0] = \Pr[g(Y) \geq g(0)] = \Pr[g(Y) \geq 1] \leq \frac{E[g^k(Y)]}{1^k} = E[g^k(Y)].
\]

On the other hand,
\[
\Pr[Y \geq 0] = \int_0^\infty f_Y(y)dy = \int_0^\infty E[g^k(Y)] \frac{f_W(y)}{g^k(y)}dy = E[g^k(Y)] \int_0^\infty \frac{f_W(y)}{g^k(y)}dy.
\]

(b) Since \(g(y)\) is a positive function, \(E[g^k(Y)] > 0\) (This must be mentioned in the proof!). Hence, by (a), we obtain that
\[
\int_0^\infty \frac{f_W(y)}{g^k(y)}dy \leq 1.
\]

The nonnegativity of the integral can be verified by noting that \(f_W(y)/g^k(y) \geq 0\) for all \(y\). \(\Box\)

3. (Section 20)

(a) Fix a probability space \((\Omega = \{0, 1, 2, 3, \cdots \}, \mathcal{F} = \{\emptyset, \Omega\}, P)\). Define a random variable \(X\) that is well-defined over this probability space. (Hint: The event \(\omega \in \Omega : X(\omega) \leq x\) must lie in \(\mathcal{F}\) for any real \(x\).)

(b) For a probability space \((\Omega = \{1, 2, 3, 4, 5, 6\}, \mathcal{F}, P)\), and a random variable satisfying \(X(1) = X(3) = X(5) = -1\) and \(X(2) = X(4) = X(6) = 1\), find the \(\sigma\)-field generated by \(X\).
Answer:

(a) $X(n) = 1$ for all $n \in \Omega$.

(b) $[\omega \in \Omega : X(\omega) \leq x] = \begin{cases} \emptyset, & x < -1; \\ \{1, 3, 5\}, & -1 \leq x < 1; \\ \Omega, & x > 1 \end{cases}$

Hence, the smallest $\sigma$-field that contains $\emptyset$, $\{1, 3, 5\}$ and $\Omega$ is $\{\emptyset, \{1, 3, 5\}, \{2, 4, 6\}, \Omega\}$.

4. (Section 21)

(a) Prove the generalized Hölder’s inequality, i.e., for $p_j > 0$ with $\sum_{j=1}^{n} 1/p_j = 1$,

$$E \left[ \prod_{j=1}^{n} |X_j| \right] \leq \prod_{j=1}^{n} E^{1/p_j} [|X_j|^{p_j}] .$$

(Note: Remember to give the condition under which equality holds.)

(b) How to modify the generalized Hölder’s inequality if $\sum_{j=1}^{n} 1/p_j \neq 1$ (but still $p_j > 0$ for all $j$).

Proof:

(a) Since the inequality is trivially valid, if $\prod_{j=1}^{n} E^{1/p_j} [|X|^{p_j}] = 0$.

Without loss of generality, assume $\prod_{j=1}^{n} E^{1/p_j} [|X|^{p_j}] > 0$.

- $\exp\{x\}$ is a convex function in $x$. Hence, by Jensen’s inequality,

$$\exp \left\{ \sum_{j=1}^{n} \frac{1}{p_j} s_j \right\} \leq \sum_{j=1}^{n} \frac{1}{p_j} \exp \{s_j\} .$$

Since $e^x$ is strictly convex, equality holds iff $s_j = s$ for all $j$.

- Let $a_j = \exp\{s_j/p_j\}$. Then the above inequality becomes:

$$\prod_{j=1}^{n} a_j \leq \sum_{j=1}^{n} \frac{1}{p_j} a_j^{p_j} ,$$

Equality holds iff $a_j^{p_j} = e^s$ for all $j$.

whose validity is not restricted to positive $a_j$ but to non-negative $a_j$. 
• By letting \( a_j = |X_j|/E^{1/p_j} [X_j^{p_j}] \), we obtain:
\[
\frac{|\prod_{j=1}^n X_j|}{\prod_{j=1}^n E^{1/p_j} [X_j^{p_j}]} \leq \sum_{j=1}^n \frac{1}{p_j} E^{1/p_j} [X_j^{p_j}].
\]

Taking the expectation values of both sides yields:
\[
\frac{E[|\prod_{j=1}^n X_j|]}{\prod_{j=1}^n E^{1/p_j} [X_j^{p_j}]} \leq \sum_{j=1}^n \frac{1}{p_j} = 1.
\]
Equality holds iff \( \Pr\left( \frac{|X_j|^{p_j}}{E[X_j^{p_j}]} = \text{constant for all } j \right) = 1. \)

(b) Suppose \( \sum_{j=1}^n 1/p_j = q > 0. \) Then \( \sum_{j=1}^n 1/(p_j q) = 1. \)
\[
E\left[ \prod_{j=1}^n X_j \right] \leq \prod_{j=1}^n E^{1/(p_j q)} [X_j^{p_j q}].
\]

5. (Section 22) Prove that equality holds for the inequality in Theorem 22.4 (slide 22-19) only when both sides are either one or zero.

**Proof:** From the proof, we found that equality holds if
\[
E\left[ S_n^2(I_{A_{n+1}} + I_{A_{n+2}} + \cdots) \right] = 0 \tag{2}
\]
\[
\sum_{k=1}^n E\left[ (S_n - S_k)^2 I_{A_k} \right] = E\left[ \sum_{k=1}^n (S_n - S_k)^2 I_{A_k} \right] = 0 \tag{3}
\]
\[
\sum_{k=1}^n E\left[ (S_k^2 - \alpha^2) I_{A_k} \right] = 0. \tag{4}
\]
Since \( S_n^2(I_{A_{n+1}} + I_{A_{n+2}} + \cdots) \) is non-negative, (2) tells that \( S_n^2(I_{A_{n+1}} + I_{A_{n+2}} + \cdots) = 0 \) with probability 1, which implies either \( \max_{1 \leq k \leq n} |S_k| \geq \alpha \) or \( S_n = 0. \) We discuss the two conditions separately below.

- The condition \( \max_{1 \leq k \leq n} |S_k| \geq \alpha \) implies that exact one of \( \{I_{A_k}\}_{k=1}^n \) is one. Let \( I_{A_j} = 1 \) for some \( 1 \leq j \leq n. \) Then, (3) and (4) indicate that \( (S_n - S_j)^2 = 0 \) and \( S_j^2 = \alpha^2, \) i.e., \( S_n = S_j \) and \( \Pr(S_j = \alpha) = \Pr(S_j = -\alpha) = 1/2 \) for some \( 1 \leq j \leq n \) satisfying
Thus, the first condition that makes the inequality in Theorem 22.4 equal is:

\[ \exists j \text{ among } 1 \leq j \leq n \text{ such that } \max_{1 \leq k \leq j-1} |S_k| < \alpha \text{ and } S_n = S_j \]

and \( \Pr(S_j = \alpha) = \Pr(S_j = -\alpha) = \frac{1}{2} \),

a case that makes both sides of the inequality one!

- The condition \( S_n = 0 \) reduces (3) and (4) to

\[ \sum_{k=1}^{n} E[S_k^2 I_{A_k}] = 0 \]

and \( \sum_{k=1}^{n} \alpha^2 E[I_{A_k}] = 0 \). Hence, \( I_{A_k} = 0 \) with probability 1 for every \( 1 \leq k \leq n \). Thus, the second situation that equates the inequality in Theorem 22.4 is

\[ S_n = 0 \text{ and } \max_{1 \leq k \leq n-1} |S_k| < \alpha, \]

a trivial case that makes both sides of the inequality zero!

6. (Section 25) Based on Theorem 25.11, give an example that \( X_n \Rightarrow X \) and \( E[|X|] < \liminf_{n \to \infty} E[|X_n|] \). (Hint: Let \( X_n = X + \delta_n Y \) for non-negative \( X \), \( Y \) and \( \delta_n \). Then define the required properties of \( X \), \( Y \) and constant sequence \( \delta_n \) to satisfy the two conditions. Remember to use “Properties regarding convergence in distribution” to prove that \( X_n \Rightarrow X \).)

**Answer:** Find non-negative \( X \) and \( Y \) such that \( E[X] < \infty \) and \( E[Y] = \infty \). Also, find a positive sequence \( \{\delta_n\}_{n=1}^{\infty} \) satisfying \( \delta_n \to 0 \).

1. \( X_n \Rightarrow X \): From slide 25-15, \( Y \Rightarrow Y \) and \( \delta_n \to 0 \) implies \( \delta_n Y \Rightarrow 0 \). From Theorem 25.4, \( X \Rightarrow X \) and \( X - (X + \delta_n Y) \Rightarrow 0 \) implies \( X + \delta_n Y \Rightarrow X \).

2. \( E[|X|] < \liminf_{n \to \infty} E[|X_n|] \):

\[
E[|X|] = E[X] < \liminf_{n \to \infty} E[|X_n|] = \liminf_{n \to \infty} E[X_n] = \liminf_{n \to \infty} (E[X] + \delta_n E[Y]) = \infty.
\]
7. (Section 26)

(a) Point out how to refine the proof on slide 26-52∼54 so that the theorem statement can be improved to:

**Theorem** Suppose the support of the distribution of random variable $X$ is contained in $[0, 2\pi]$. Then

$$\frac{1}{2} \Pr[X = a] + \Pr[a < X < b] + \frac{1}{2} \Pr[X = b] = \lim_{m \to \infty} \int_{a}^{b} \sigma_m(t) dt,$$

if $0 < a < b < 2\pi$, where

$$\sigma_m(t) = \frac{1}{2\pi m} \int_{0}^{2\pi} \frac{\sin^2[m(x - t)/2]}{\sin^2[(x - t)/2]} dF_X(x).$$

(Hint: $\sin^2(ms/2)/\sin^2(s/2)$ is an even function in $s$.)

(b) Based on the above theorem, we have the next corollary.

**Corollary** Suppose the support of the distribution of random variable $X$ is contained in $[-\pi, \pi]$. Let $f_X(x)$ be the density of $X$. Then

$$\int_{a}^{b} f_X(x) dx = \lim_{m \to \infty} \int_{a}^{b} \sigma_m(t) dt,$$

if $-\pi < a < b < \pi$, where $\sigma_m(t) = \frac{1}{2\pi m} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^{\ell} c_k e^{-ikt}$, $c_m = \varphi_X(m)$, and $\varphi_X(t) = \int_{0}^{2\pi} e^{itx} dF_X(x)$.

Now let a (possibly complex) function $h(t)$ have non-negative bounded real spectrum $H(f) = \int_{-\infty}^{\infty} h(t) e^{-itf} dt$ satisfying $H(f) = 0$ for $|f| > \pi$. By the above corollary, represent $\int_{a}^{b} H(f) df$ in terms of the samples of $h(t)$, where $-\pi < a < b < \pi$. (Note: No integration operation is allowed to leave unresolved in the answer.)

(c) In (b), if we further assume that $H(f)$ is symmetric (i.e., $H(f) = H(-f)$; hence, $h(t)$ is real), then show that $\int_{a}^{b} H(f) df$ can be re-formulated in the form of sinc function.

**Proof:**

7
(a) Since \( \sin^2(\frac{ms}{2})/\sin^2(s/2) \) is a bounded even function,

\[
\frac{1}{2\pi m} \int_{-\pi}^{0} \frac{\sin^2(\frac{ms}{2})}{\sin^2(s/2)} \, ds = \frac{1}{2\pi m} \int_{0}^{\pi} \frac{\sin^2(\frac{ms}{2})}{\sin^2(s/2)} \, ds.
\]

Hence,

\[
\frac{1}{2\pi m} \int_{-\pi}^{0} \frac{\sin^2(\frac{ms}{2})}{\sin^2(s/2)} \, ds = \frac{1}{2\pi m} \int_{0}^{\pi} \frac{\sin^2(\frac{ms}{2})}{\sin^2(s/2)} \, ds = \frac{1}{2},
\]

which implies that

\[
\lim_{m \to \infty} \frac{1}{2\pi m} \left( \int_{a-x}^{b-x} \frac{\sin^2(\frac{ms}{2})}{\sin^2(s/2)} \, ds \right) = \begin{cases} 
0, & \text{if } a-x > 0; \\
\frac{1}{2}, & \text{if } a-x = 0; \\
1, & \text{if } a-x < 0 < b-x; \\
\frac{1}{2}, & \text{if } b-x = 0; \\
0, & \text{if } b-x < 0
\end{cases}
\]

Accordingly, the new theorem statement holds.

(b) Let the density of \( X \) in (a) be \( f_X(x) = H(x)/\int_{-\pi}^{\pi} H(f) \, df \). Then,

\[
h(t) = \int_{-\infty}^{\infty} H(f) e^{i2\pi ft} \, df = \left( \int_{-\pi}^{\pi} H(f) \, df \right) \left( \int_{-\infty}^{\infty} f_X(x) e^{i2\pi tx} \, dx \right) = \left( \int_{-\pi}^{\pi} H(f) \, df \right) \varphi_X(2\pi t),
\]

which implies

\[
c_m = \varphi_X(m) = h(m/(2\pi))/\left( \int_{-\pi}^{\pi} H(f) \, df \right).
\]

Hence,

\[
\sigma_m(t) = \frac{1}{2\pi m} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^{\ell} c_k e^{-itk} = \frac{1}{2\pi m \left( \int_{-\pi}^{\pi} H(f) \, df \right)} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^{\ell} h\left( \frac{k}{2\pi} \right) e^{-itk}
\]
and

\[
\frac{\int_a^b H(f) df}{\int_{-\pi}^\pi H(f) df} = \lim_{m \to \infty} \int_a^b \sigma_m(t) dt
\]

\[
= \frac{1}{2\pi (\int_{-\pi}^\pi H(f) df)} \lim_{m \to \infty} \frac{1}{m} \int_a^b \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^k h \left( \frac{k}{2\pi} \right) e^{-ikt} dt
\]

\[
= \frac{1}{2\pi (\int_{-\pi}^\pi H(f) df)} \lim_{m \to \infty} \frac{1}{m} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^k h \left( \frac{k}{2\pi} \right) \int_a^b e^{-ikt} dt
\]

\[
= \frac{1}{2\pi (\int_{-\pi}^\pi H(f) df)} \lim_{m \to \infty} \frac{1}{m} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^k h \left( \frac{k}{2\pi} \right) \frac{e^{-ika} - e^{-ikb}}{ik}.
\]

Consequently,

\[
\int_a^b H(f) df = \lim_{m \to \infty} \frac{1}{2\pi m} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^k h \left( \frac{k}{2\pi} \right) \frac{e^{-ika} - e^{-ikb}}{ik}.
\]

(c) Observe that

\[
\int_{-b}^b H(f) df = \lim_{m \to \infty} \frac{1}{2\pi m} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^k h \left( \frac{k}{2\pi} \right) \frac{e^{ikb} - e^{-ikb}}{ik}
\]

\[
= \lim_{m \to \infty} \frac{1}{m} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^k h \left( \frac{k}{2\pi} \right) \frac{\sin(kb)}{\pi k}.
\]

Then the problem can be solved by noting that

\[
2 \int_a^b H(f) df = \begin{cases} 
\int_{-b}^b H(f) df - \int_a^a H(f) df, & 0 < a < b; \\
\int_{-b}^b H(f) df + \int_a^a H(f) df, & a < 0 < b; \\
\int_{-b}^b H(f) df - \int_a^a H(f) df, & a < b < 0;
\end{cases}
\]

8. (Section 27) Show that if \(z_1, \ldots, z_m\) and \(w_1, \ldots, w_m\) are complex numbers, not necessarily of modulus at most 1, then

\[
|z_1 \times z_2 \times \cdots \times z_m - w_1 \times w_2 \times \cdots \times w_m| \leq \sum_{k=1}^m |z_k - w_k|
\]
may not be true. (Hint: Give an example of $z_i$ and $w_i$ that are not of modulus at most 1, and show that the inequality does not hold.)

**Answer:** Take $m = 2$, we can find that in the proof, only the last step requires the condition of “modulus at most 1”. Specifically,

$$|z_1 \times z_2 - w_1 \times w_2| = \left|(z_1 - w_1)z_2 + w_1(z_2 - w_2)\right| \leq |(z_1 - w_1)z_2| + |w_1(z_2 - w_2)| = |z_1 - w_1||z_2| + |w_1||z_2 - w_2| \leq |z_1 - w_1| + |z_2 - w_2|.$$

A counterexample is easy to construct by taking $w_i = 0$ for all $i$, and $|z_1 \times z_2 \times \cdots \times z_m| > \sum_{k=1}^{m} |z_k|$, e.g., $z_k = k$ for all $k$.

9. (In general)

(a) Prove that $E[X] = \int_0^{\infty} \Pr[X > t]dt$ for non-negative bounded random variable $X$ with density $f(x)$. (Hint: Fubini’s Theorem.)

(b) Using (a) to prove that $E[X] = \int_0^{\infty} \Pr[X > t]dt$ for non-negative (possibly unbounded) random variable $X$ with density $f(x)$. (Hint: Define $Y_n = X I_{[x \leq n]}$, and Fatou’s lemma tells that

$$E[X] \leq \liminf_{n \to \infty} E[Y_n] \quad \text{and} \quad \int_0^{\infty} \Pr[X > t]dt \leq \liminf_{n \to \infty} \int_0^{\infty} \Pr[Y_n > t]dt.$$

**Proof:**

(a) Denote by $M$ the bound for random variable $X$, i.e., $\Pr[X \leq M] = 1$. By Fubini’s Theorem (the condition for Fubini’s Theorem is always valid for bounded integration region),

$$\int_0^{\infty} \Pr[X > t]dt = \int_0^{M} \Pr[X > t]dt = \int_0^{M} \left(\int_t^{M} f(x)dx\right) dt = \int_0^{M} \left(\int_0^{x} f(x)dt\right) dx = \int_0^{M} xf(x)dx = E[X].$$
Define \( Y_n = X I_{[X \leq n]} \). Then \( \Pr[Y_n > t] \to \Pr[X > t] \) as \( n \to \infty \), or equivalently, \( Y_n \Rightarrow X \).

By Fatou’s Lemma,

\[
E[Y_n] \leq E[X] \leq \liminf_{n \to \infty} E[Y_n].
\]

Therefore, \( E[X] = \lim_{n \to \infty} E[Y_n] \).

On the other hand,

\[
\int_0^\infty \Pr[Y_n > t]dt \leq \int_0^\infty \Pr[X > t]dt \leq \liminf_{n \to \infty} \int_0^\infty \Pr[Y_n > t]dt,
\]

which implies that

\[
\int_0^\infty \Pr[X > t]dt = \lim_{n \to \infty} \int_0^\infty \Pr[Y_n > t]dt.
\]

The proof is completed by noting that \( E[Y_n] = \int_0^\infty \Pr[Y_n > t]dt \) from (a).

10. (In general) A random variable has a **lattice distribution** if for some \( a \) and \( b \), where \( b \neq 0 \), the set \( \{ a + nb : n = 0, \pm 1, \pm 2, \ldots \} \) supports the distribution of \( X \).

(a) Prove that \( X \) has a lattice distribution, if its characteristic function \( |\varphi_X(t)| = 1 \) for some \( t \neq 0 \).

(Hint: A complex value \( \varphi_X(t_0) \) satisfying \( |\varphi_X(t_0)| = 1 \) can be represented as \( \varphi_X(t_0) = e^{it_0a} \) for some \( a \). That a non-negative function like \( [1 - \cos(it_0(x - a))] \) has integral zero implies that the function is exactly zero in the integration domain.)

(b) Prove that if \( X \) has a lattice distribution, then its characteristic function \( |\varphi_X(t)| = 1 \) for some \( t \neq 0 \).

(Hint: Let \( b = 2\pi/t_0 \).)

**Proof:**
(a) If $|\varphi_X(t_0)| = 1$ for some $t_0 \neq 0$, then $\varphi_X(t_0) = e^{it_0a}$ for some $a$.

\[
0 = 1 - e^{-it_0a}\varphi_X(t_0) = \int_{-\infty}^{\infty} dF_X(x) - e^{-it_0a} \int_{-\infty}^{\infty} e^{it_0x} dF_X(x) = \int_{-\infty}^{\infty} (1 - e^{it_0(x-a)}) dF_X(x) \\
= \int_{-\infty}^{\infty} [1 - \cos(it_0(x-a))] dF_X(x) + i \int_{-\infty}^{\infty} \sin(it_0(x-a)) dF_X(x).
\]

As $[1 - \cos(it_0(x-a))] \geq 0$ and $\int_{-\infty}^{\infty} [1 - \cos(it_0(x-a))] dF_X(x) = 0$, we conclude that

\[
\Pr[\cos(it_0(X-a)) = 1] = 1,
\]

which implies that

\[
\Pr[t_0(X-a) = 2n\pi \text{ for } n = 0, \pm 1, \pm 2, \ldots] = 1.
\]

(b) Let $b = 2\pi/t_0$.

\[
\varphi_X(t_0) = \sum_{n=0,\pm 1,\pm 2,\ldots} e^{it_0(a+nb)} p_n = \sum_{n=0,\pm 1,\pm 2,\ldots} e^{it_0(a+2\pi n/t_0)} p_n = e^{it_0a} \sum_{n=0,\pm 1,\pm 2,\ldots} e^{2\pi n} p_n = e^{it_0a}.
\]

So $|\varphi_X(t_0)| = 1$. \hfill \Box