Berry-Esseen Theorem

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Historical aspects

• The central limit theorem (CLT) concerns the situation that the limit distribution of the normalized sum is normal.

• As an example, for i.i.d. zero-mean sequence $X_1, X_2, \ldots$,

$$\frac{X_1 + \cdots + X_n}{\sqrt{nE[X_1^2]}} \Rightarrow N,$$

where $N$ has standard normal distribution.

• **Question:** What is the rate of convergence of normalized sum distribution to standard normal distribution?
Historical aspects

• The first convergence rate estimates in the CLT were obtained by A. M. Lyapounov in 1900-1901.

• In the beginning of 1940s, the classic Berry-Esseen estimate came to the light:

\[
\sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)| \leq C \frac{\beta_3}{\sigma^3 \sqrt{n}},
\]

where

- \( F_n \) is the cdf of \( (X_1 + \cdots + X_n)/\sqrt{nE[X_1^2]} \),
- \( \Phi \) is the standard normal cdf,
- \( \beta_3 = E \left[ |X - E[X]|^3 \right] \),
- \( \sigma^2 = E \left[ |X - E[X]|^2 \right] \), and
- \( C \) is a universal constant, independent of \( n \).

• In fact, it was due to Lyapounov that the finiteness of \( (2 + \delta) \)th absolute moment, where \( 0 < \delta < 1 \), guarantees

\[
\sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)| = O(n^{-\delta/2}) \text{ as } n \to \infty.
\]
Berry-Esseen theorem

- Classic Berry-Esseen theorem

  - For independent zero-mean random variables \( \{X_i\}_{i=1}^n \),

    \[
    \sup_{a \in \mathbb{R}} \left| \Pr \left[ \frac{1}{s_n} (X_1 + \cdots + X_n) \leq a \right] - \Phi(a) \right| \leq C \frac{r_n}{s_n^3},
    \]

    where
    
    \* \( s_n^2 \equiv \text{sum of the marginal variances} \),
    
    \* \( r_n \equiv \text{sum of the marginal absolute 3rd central moments} \),
    
    \* \( C \equiv \text{absolute constant} \), and
    
    \* \( \Phi(\cdot) \equiv \text{the unit Gaussian cumulative distribution function (cdf)} \).

- Notably, only the first three moments are involved in the Berry-Esseen inequality.
Berry-Esseen theorem

\[ \sup_{a \in \mathbb{R}} \left| \Pr \left[ \frac{1}{s_n} (X_1 + \cdots + X_n) \leq a \right] - \Phi(a) \right| \leq C \frac{\sigma_n}{s_n^3}. \]

- (Feller’s book, 1979)
  - \( C = 6 \) for an independent sample sum.
  - \( C = 3 \) for an independent and identically distributed sample sum.

- (Beek 1972) \( C = 0.7975 \) for an independent sample sum.

- (Shiganov 1986)
  - \( C = 0.7915 \) for an independent sample sum.
  - \( C = 0.7655 \) for an independent and identically distributed sample sum.
Berry-Esseen theorem

- Again, the remarkable aspect of the Berry-Esseen theorem is that the upper bound depends only on the variance and the 3rd central moment.

- Hence, it can provide a good probability estimate based on the first three moments.
Berry-Esseen theorem and probability bound

Technique (behind the proof): Pass the difference through a bandlimited filter.

**Lemma** Fix a symmetric bandlimited filtering function

\[ v_T(x) = \frac{1 - \cos(Tx)}{\pi T x^2} = \frac{2\sin^2(Tx/2)}{\pi T x^2} \]

with characteristic function

\[ \omega_T(\zeta) \triangleq \int_{-\infty}^{\infty} v_T(x)e^{-j\zeta x}dx = \begin{cases} 
1 - \frac{|\zeta|}{T}, & \text{if } |\zeta| \leq T; \\
0, & \text{otherwise}.
\end{cases} \]

For any cdf \( H(\cdot) \) on the real line \( \mathbb{R} \),

\[
\sup_{x \in \mathbb{R}} |\Delta_T(x)| \geq \frac{1}{2} \eta - \frac{6}{T \pi \sqrt{2\pi}} h \left( \frac{T \sqrt{2\pi}}{2} \eta \right),
\]

where \( \eta \triangleq \sup_{x \in \mathbb{R}} |H(x) - \Phi(x)| \), \( \Delta_T(t) \triangleq \int_{-\infty}^{\infty} [H(t - x) - \Phi(t - x)] \times v_T(x)dx \),

and \( h(u) \triangleq \pi u \int_{0}^{\infty} \frac{v_T \left( \frac{t}{T} \right) dt}{T} = \frac{\pi}{2} u + 1 - \cos(u) - u \int_{0}^{u} \frac{\sin(x)}{x} dx \).
Summary: \( \eta \) is the desired difference between any cdf \( H(\cdot) \) and standard normal cdf \( \Phi(\cdot) \).

\[
\sup_{x \in \mathbb{R}} \left( [H(x) - \Phi(x)] * v_T(x) \right) \\
\geq \frac{1}{2} \eta - \frac{6}{T \pi \sqrt{2\pi}} \cdot \pi \cdot \frac{T \sqrt{2\pi}}{2} \eta \int_{T \sqrt{2\pi \eta/2}}^{\infty} v_T \left( \frac{t}{T} \right) \frac{dt}{T} \\
= \frac{1}{2} \eta - 3 \eta \int_{\sqrt{2\pi \eta/2}}^{\infty} v_T(y) dy \\
= \eta \left( \frac{1}{2} - 3 \int_{\sqrt{2\pi \eta/2}}^{\infty} v_T(y) dy \right) \\
= \left( \sup_{x \in \mathbb{R}} |H(x) - \Phi(x)| \right) \left( \frac{1}{2} - 3 \int_{\sqrt{2\pi \eta/2}}^{\infty} v_T(y) dy \right)
\]

Observation

- The maximum filter output bounds the maximum absolute value of filter input.
Berry-Esseen theorem and probability bound

Proof:

• The right-continuity of the cdf $H(\cdot)$ and the continuity of the Gaussian unit cdf $\Phi(\cdot)$ together indicate the right-continuity of $|H(x) - \Phi(x)|$, which in turn implies the existence of $x_0 \in \mathbb{R}$ satisfying

  
  either \( \eta = |H(x_0) - \Phi(x_0)| \) or \( \eta = \lim_{x \uparrow x_0} |H(x) - \Phi(x)| > |H(x_0) - \Phi(x_0)| \).

• We then distinguish between three cases:

  Case A) \( \eta = H(x_0) - \Phi(x_0) \);
  Case B) \( \eta = \Phi(x_0) - H(x_0) \);
  Case C) \( \eta = \lim_{x \uparrow x_0} |H(x) - \Phi(x)| > |H(x_0) - \Phi(x_0)| \).
Berry-Esseen theorem and probability bound

- **Case A)** \( \eta = H(x_0) - \Phi(x_0) \).

In this case, we note that for \( s > 0 \),

\[
H(x_0 + s) - \Phi(x_0 + s) \geq H(x_0) - \left[ \Phi(x_0) + \frac{s}{\sqrt{2\pi}} \right] \tag{1}
\]

\[
= \eta - \frac{s}{\sqrt{2\pi}}, \tag{2}
\]

where (1) follows from \( \sup_{x \in \mathbb{R}} |\Phi'(x)| = 1/\sqrt{2\pi} \).

Observe that (2) implies

\[
H \left( x_0 + \frac{\sqrt{2\pi}}{2} \eta - x \right) - \Phi \left( x_0 + \frac{\sqrt{2\pi}}{2} \eta - x \right) \geq \eta - \frac{1}{\sqrt{2\pi}} \left( \frac{\sqrt{2\pi}}{2} \eta - x \right)
\]

\[
= \frac{1}{2} \eta + \frac{x}{\sqrt{2\pi}},
\]

for \( |x| < \eta \sqrt{2\pi}/2 \).
Together with the fact that $H(x) - \Phi(x) \geq -\eta$ for all $x \in \mathbb{R}$, we obtain

$$\sup_{x \in \mathbb{R}} |\Delta_T(x)| \geq \Delta_T \left( x_0 + \frac{\sqrt{2\pi}}{2} \eta \right)$$

$$= \int_{-\infty}^{\infty} \left[ H \left( x_0 + \frac{\sqrt{2\pi}}{2} \eta - x \right) - \Phi \left( x_0 + \frac{\sqrt{2\pi}}{2} \eta - x \right) \right] \times v_T(x) dx$$

$$= \int_{|x| < \eta \sqrt{2\pi}/2} \left[ H \left( x_0 + \frac{\sqrt{2\pi}}{2} \eta - x \right) - \Phi \left( x_0 + \frac{\sqrt{2\pi}}{2} \eta - x \right) \right] \times v_T(x) dx$$

$$+ \int_{|x| \geq \eta \sqrt{2\pi}/2} \left[ H \left( x_0 + \frac{\sqrt{2\pi}}{2} \eta - x \right) - \Phi \left( x_0 + \frac{\sqrt{2\pi}}{2} \eta - x \right) \right] \times v_T(x) dx$$

$$\geq \int_{|x| < \eta \sqrt{2\pi}/2} \left[ \frac{1}{2} \eta + \frac{x}{\sqrt{2\pi}} \right] \times v_T(x) dx + \int_{|x| \geq \eta \sqrt{2\pi}/2} (-\eta) \times v_T(x) dx$$

$$= \int_{|x| < \eta \sqrt{2\pi}/2} \frac{1}{2} \eta \times v_T(x) dx + \int_{|x| \geq \eta \sqrt{2\pi}/2} (-\eta) \times v_T(x) dx,$$

where the last equality holds because of the symmetry of the filtering function $v_T(\cdot)$. 

Berry-Esseen theorem and probability bound
The quantity of \( \int_{|x| \geq \eta \sqrt{2\pi}/2} v_T(x) dx \) can be derived as follows:

\[
\int_{|x| \geq \eta \sqrt{2\pi}/2} v_T(x) dx = 2 \int_{\eta \sqrt{2\pi}/2}^{\infty} v_T(x) dx \\
= 2 \int_{\eta T \sqrt{2\pi}/2}^{\infty} v_T \left( \frac{u}{T} \right) \frac{du}{T} \\
= \frac{4}{\eta T \pi \sqrt{2\pi}} h \left( \frac{T \sqrt{2\pi}}{2} \eta \right).
\]

Continuing from (3),

\[
\sup_{x \in \mathbb{R}} |\Delta_T(x)| \geq \frac{1}{2} \eta \left[ 1 - \frac{4}{\eta T \pi \sqrt{2\pi}} h \left( \frac{T \sqrt{2\pi}}{2} \eta \right) \right] \\
- \eta \cdot \left[ \frac{4}{\eta T \pi \sqrt{2\pi}} h \left( \frac{T \sqrt{2\pi}}{2} \eta \right) \right] \\
= \frac{1}{2} \eta - \frac{6}{T \pi \sqrt{2\pi}} h \left( \frac{T \sqrt{2\pi}}{2} \eta \right). \]
Berry-Esseen theorem and probability bound

Case B) \( \eta = \Phi(x_0) - H(x_0) \).

Similar to Case A), we first derive for \( s > 0 \),

\[
\Phi(x_0 - s) - H(x_0 - s) \geq \left[ \Phi(x_0) - \frac{s}{\sqrt{2\pi}} \right] - H(x_0) = \eta - \frac{s}{\sqrt{2\pi}},
\]

and then obtain

\[
\Phi \left( x_0 - \frac{\sqrt{2\pi}}{2} \eta - x \right) - H \left( x_0 - \frac{\sqrt{2\pi}}{2} \eta - x \right) \geq \eta - \frac{1}{\sqrt{2\pi}} \left( \frac{\sqrt{2\pi}}{2} \eta + x \right)
\]

\[
= \frac{1}{2} \eta - \frac{x}{\sqrt{2\pi}},
\]

for \( |x| < \eta\sqrt{2\pi}/2 \).
Together with the fact that \( \Phi(x) - H(x) \geq -\eta \) for all \( x \in \mathbb{R} \), we obtain

\[
\sup_{x \in \mathbb{R}} |\Delta_T(x)| \geq -\Delta_T \left( x_0 - \frac{\sqrt{2\pi}}{2} \eta \right)
\]

\[
\geq \int_{|x| < \eta \sqrt{2\pi}/2} \left[ \frac{1}{2} \eta - \frac{x}{\sqrt{2\pi}} \right] \times v_T(x) dx
\]

\[
+ \int_{|x| \geq \eta \sqrt{2\pi}/2} (-\eta) \times v_T(x) dx
\]

\[
= \int_{|x| < \eta \sqrt{2\pi}/2} \frac{1}{2} \eta \times v_T(x) dx + \int_{|x| \geq \eta \sqrt{2\pi}/2} (-\eta) \times v_T(x) dx
\]

\[
= \frac{1}{2} \eta - \frac{6}{T\pi \sqrt{2\pi}} \ h \left( \frac{T \sqrt{2\pi}}{2} \eta \right).
\]
Berry-Esseen theorem and probability bound

Case C) $\eta = \lim_{x \uparrow x_0} |H(x) - \Phi(x)| > |H(x_0) - \Phi(x_0)| \geq 0$.

In this case, we observe that for any $0 < \delta < \eta$, there exists $x'_0$ such that $|H(x'_0) - \Phi(x'_0)| \geq \eta - \delta \triangleq \eta'$. We can then follow the procedure of the previous two cases to obtain:

$$\sup_{x \in \mathbb{R}} |\Delta_T(x)| \geq \frac{1}{2} \eta' - \frac{6}{T\pi \sqrt{2\pi}} h\left(\frac{T \sqrt{2\pi}}{2 \eta'}\right).$$

The proof is completed by noting that $\eta'$ can be made arbitrarily close to $\eta$. $\square$
Lemma For any cumulative distribution function $H(\cdot)$ with zero-mean and unit variance, its characteristic function $\varphi_H(\zeta)$ satisfies that

$$\eta \leq \frac{1}{\pi} \int_{-T}^{T} \left| \varphi_{H}(\zeta) - e^{-(1/2)\zeta^2} \right| \frac{d\zeta}{|\zeta|} + \frac{12}{T\pi \sqrt{2\pi}} h \left( \frac{T \sqrt{2\pi} \eta}{2} \right),$$

where $\eta$ and $h(\cdot)$ are defined in the previous lemma.

This lemma transforms the bound in the previous lemma into frequency domain, namely the characteristic function domain.
Berry-Esseen theorem and probability bound

Proof:

• Observe that

$$\Delta_T(t) = \int_{-\infty}^{\infty} [H(t - x) - \Phi(t - x)] \times v_T(x) dx$$

is nothing but a convolution of $v_T(\cdot)$ and $H(\cdot) - \Phi(\cdot)$.

• By Fourier inversion theorem,

$$\frac{d}{dt}(\Delta_T(t)) \bigg|_{t=x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-j\xi x}}{1} \left[ \varphi_H(\xi) - e^{-(1/2)e^\xi^2} \right] \omega_T(\xi) d\xi$$

where the second step follows from bandlimit assumption.

Notably, $\varphi_H(\xi)$ and $e^{-\xi^2/2}$ are the Fourier transforms with respect to measures $dH(x)$ and $d\Phi(x)$, not the Fourier transforms of $H(x)$ and $\Phi(x)$.

Actually, $H(x)$ and $\Phi(x)$ have no Fourier transforms.
Berry-Esseen theorem and probability bound

- Integrating with respect to $x$, we obtain

$$
\Delta_T(x) = \frac{1}{2\pi} \int_{-T}^{T} e^{-j\xi x} \left[ \varphi_H(\xi) - e^{-(1/2)\xi^2} \right] \omega_T(\xi) d\xi,
$$

where no integration constant appears since both sides go to zero as $|x| \to \infty$.

Notably,

$$
f'(x) = g'(x) \Rightarrow f(x) = g(x) + c,
$$

where $c$ is the integration constant.
Berry-Esseen theorem and probability bound

Accordingly,

\[
\sup_{x \in \mathbb{R}} |\Delta_T(x)| = \sup_{x \in \mathbb{R}} \frac{1}{2\pi} \left| \int_{-T}^{T} e^{-j\xi x} \frac{\left[ \varphi_H(\xi) - e^{-(1/2)\xi^2} \right]}{-j\xi} \omega_T(\xi) d\xi \right|
\]

\[
\leq \sup_{x \in \mathbb{R}} \frac{1}{2\pi} \int_{-T}^{T} \left| e^{-j\xi x} \frac{\left[ \varphi_H(\xi) - e^{-(1/2)\xi^2} \right]}{-j\xi} \omega_T(\xi) \right| d\xi
\]

\[
= \sup_{x \in \mathbb{R}} \frac{1}{2\pi} \int_{-T}^{T} \left| \varphi_H(\xi) - e^{-(1/2)\xi^2} \right| \cdot |\omega_T(\xi)| \frac{d\xi}{|\xi|}
\]

\[
\leq \sup_{x \in \mathbb{R}} \frac{1}{2\pi} \int_{-T}^{T} \left| \varphi_H(\xi) - e^{-(1/2)\xi^2} \right| \frac{d\xi}{|\xi|}
\]

\[
= \frac{1}{2\pi} \int_{-T}^{T} \left| \varphi_H(\xi) - e^{-(1/2)\xi^2} \right| \frac{d\xi}{|\xi|}.
\]
Berry-Esseen theorem and probability bound

Together with

$$\sup_{x \in \mathbb{R}} |\Delta_T(x)| \geq \frac{1}{2} \eta - \frac{6}{T\pi \sqrt{2\pi}} h \left( \frac{T\sqrt{2\pi}}{2} \eta \right),$$

we finally have

$$\eta \leq \frac{1}{\pi} \int_{-T}^{T} \left| \varphi_H(\zeta) - e^{-(1/2)\zeta^2} \right| \frac{d\zeta}{|\zeta|} + \frac{12}{T\pi \sqrt{2\pi}} h \left( \frac{T\sqrt{2\pi}}{2} \eta \right).$$
Theorem Let $Y_n = \sum_{i=1}^{n} X_i$ be sum of i.i.d. random variables. Assume $n \geq 3$. Denote the mean and variance of $X_1$ by $\mu$ and $\sigma^2$, respectively. Define

$$\hat{\rho} \triangleq E \left[ |X_1 - \mu|^3 \right].$$

Also denote the cdf of $(Y_n - E[Y_n])/(\sigma \sqrt{n})$ by $H_n(\cdot)$. Then

$$\sup_{y \in \mathbb{R}} |H_n(y) - \Phi(y)| \leq 5 \frac{\hat{\rho}}{\sigma^3 \sqrt{n}}.$$
Berry-Esseen theorem and probability bound

Proof:

\[ \pi \cdot \eta \leq \int_{-T}^{T} \left| \hat{\phi}^n \left( \frac{\zeta}{\sigma \sqrt{n}} \right) - e^{-\zeta^2/2} \right| \frac{d\zeta}{|\zeta|} + \frac{12}{T \sqrt{2\pi}} h \left( \frac{T \sqrt{2\pi}}{2} \eta \right), \]

where \( \hat{\phi}(\cdot) \) is the characteristic function of \( (X_1 - \hat{\mu}) \).

**Lemma** For any complex numbers \( \alpha \) and \( \beta \),

\[ |\alpha^n - \beta^n| \leq n|\alpha - \beta|\gamma^{n-1}, \]

where \( \gamma \geq \max\{|\alpha|, |\beta|\} \).

**Proof:**

\[ |\alpha^n - \beta^n| = \left| \left( \frac{\alpha}{r} \right)^n - \left( \frac{\beta}{r} \right)^n \right| r^n \leq n \left| \frac{\alpha}{r} - \frac{\beta}{r} \right| r^n = n|\alpha - \beta| r^{n-1}. \]
Observe that the integrand satisfies
\[ \left| \hat{\varphi}^{n} \left( \frac{\zeta}{\hat{\sigma} \sqrt{n}} \right) - e^{-\zeta^2/2} \right| \]
\[ \leq n \left| \varphi \left( \frac{\zeta}{\hat{\sigma} \sqrt{n}} \right) - e^{-\zeta^2/(2n)} \right| \gamma^{n-1}, \]  
(5)
\[ \leq n \left( \left| \varphi \left( \frac{\zeta}{\hat{\sigma} \sqrt{n}} \right) - 1 + \frac{\zeta^2}{2n} \right| + \left| 1 - \frac{\zeta^2}{2n} - e^{-\zeta^2/(2n)} \right| \right) \gamma^{n-1} \]  
(6)
where the quantity \( \gamma \) in (5) requires that
\[ \left| \hat{\varphi} \left( \frac{\zeta}{\hat{\sigma} \sqrt{n}} \right) \right| \leq \gamma \]  
and
\[ \left| e^{-\zeta^2/(2n)} \right| \leq \gamma. \]
We upperbound the first and second terms in the parentheses of (6) respectively by
\[ \left| \varphi \left( \frac{\zeta}{\hat{\sigma} \sqrt{n}} \right) - 1 + \frac{\zeta^2}{2n} \right| \leq \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} |\zeta|^3 \]  
and
\[ \left| 1 - \frac{\zeta^2}{2n} - e^{-\zeta^2/(2n)} \right| \leq \frac{1}{8n^2 \zeta^4}. \]
Continuing the derivation of (6),
\[ \left| \hat{\varphi}^{n} \left( \frac{\zeta}{\hat{\sigma} \sqrt{n}} \right) - e^{-\zeta^2/2} \right| \leq n \left( \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} |\zeta|^3 + \frac{1}{8n^2 \zeta^4} \right) \gamma^{n-1}. \]  
(7)
It remains to choose \( \gamma \) that bounds both \( |\hat{\varphi}(\zeta/(\hat{\sigma} \sqrt{n}))| \) and \( \exp\{-\zeta^2/(2n)\} \) from above.
Berry-Esseen theorem and probability bound

For complex number \( z \) and reals \( b \) and \( c \),

\[
|z - b| \leq c \Rightarrow |z| \leq |b| + c.
\]

Accordingly,

\[
\left| \hat{\phi} \left( \frac{\zeta}{\hat{\sigma} \sqrt{n}} \right) - 1 + \frac{\zeta^2}{2n} \right| \leq \frac{\hat{\rho}}{6 \hat{\sigma}^3 n^{3/2}} |\zeta|^3 \Rightarrow \left| \hat{\phi} \left( \frac{\zeta}{\hat{\sigma} \sqrt{n}} \right) \right| \leq \left| 1 - \frac{\zeta^2}{2n} \right| + \frac{\hat{\rho}}{6 \hat{\sigma}^3 n^{3/2}} |\zeta|^3.
\]

• Next,

\[
\left| \hat{\phi} \left( \frac{\zeta}{\hat{\sigma} \sqrt{n}} \right) \right| \leq 1 - \frac{\zeta^2}{2n} + \frac{\hat{\rho}}{6 \hat{\sigma}^3 n^{3/2}} |\zeta|^3, \text{ if } \frac{\zeta^2}{2n} \leq 1. \tag{8}
\]

For those \( \zeta \in [-T, T] \) (which is exactly the range of integration operation in (4)), we can guarantee the validity of the condition in (8) by defining

\[
T \triangleq \hat{\sigma}^3 \sqrt{n} \left( \frac{\sqrt{2n} - 3}{\sqrt{2n} - 3} \right),
\]

and obtain

\[
\frac{\zeta^2}{2n} \leq \frac{T^2}{2n} = \frac{\hat{\sigma}^6}{2 \hat{\rho}^2} \left( \frac{\sqrt{2n} - 3}{n - 1} \right)^2 \leq \frac{1}{2} \left( \frac{\sqrt{2n} - 3}{n - 1} \right)^2 \leq 1,
\]

for \( n \geq 3 \).
Berry-Esseen theorem and probability bound

Hence, for $|\zeta| \leq T$,

$$
\left| \hat{\varphi} \left( \frac{\zeta}{\hat{\sigma} \sqrt{n}} \right) \right| \leq 1 + \left( -\frac{\zeta^2}{2n} + \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} |\zeta^3| \right)
\leq \exp \left\{ -\frac{\zeta^2}{2n} + \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} |\zeta^3| \right\}
\leq \exp \left\{ -\frac{1}{2n} \zeta^2 + \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} T \zeta^2 \right\}
\leq \exp \left\{ -\left( \frac{1}{2n} - \frac{\hat{\rho} T}{6\hat{\sigma}^3 n^{3/2}} \right) \zeta^2 \right\}
\leq \exp \left\{ -\frac{(3 - \sqrt{2})}{6(n - 1)} \zeta^2 \right\}.
$$

We can then choose

$$
\gamma \triangleq \exp \left\{ -\frac{(3 - \sqrt{2})}{6(n - 1)} \zeta^2 \right\}.
$$

Note that the above selected $\gamma$ is an upper bound of $\exp \left\{ -\zeta^2/(2n) \right\}$ for $n \geq 3/\sqrt{2} \approx 2.12$. 


Berry-Esseen theorem and probability bound

- By taking the chosen $\gamma$ into (7), the integration part in (4) becomes

$$
\int_{-T}^{T} \left| \phi^n \left( \frac{\zeta}{\hat{\sigma} \sqrt{n}} \right) - e^{-\zeta^2/2} \right| \frac{d\zeta}{|\zeta|}
\leq \int_{-T}^{T} n \left( \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} \zeta^2 + \frac{1}{8n^2} |\zeta|^3 \right) \cdot \exp \left\{ -\frac{(3 - \sqrt{2})}{6} \zeta^2 \right\} d\zeta
\leq \int_{-\infty}^{\infty} \left( \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} \zeta^2 + \frac{1}{8n} |\zeta|^3 \right) \cdot \exp \left\{ -\frac{(3 - \sqrt{2})}{6} \zeta^2 \right\} d\zeta
= \frac{\hat{\rho}}{\hat{\sigma}^3 \sqrt{n}} \left( \frac{\sqrt{6\pi}}{2 \left( 3 - \sqrt{2} \right)^{3/2}} + \frac{9}{2 \left( 3 - \sqrt{2} \right)^2 \hat{\rho} \sqrt{n}} \right)
\leq \frac{\hat{\rho}}{\hat{\sigma}^3 \sqrt{n}} \left( \frac{\sqrt{6\pi}}{2 \left( 3 - \sqrt{2} \right)^{3/2}} + \frac{9}{2 \left( 3 - \sqrt{2} \right)^2 \sqrt{n}} \right)
= \frac{1}{T} \left( \frac{\sqrt{2}n - 3}{n - 1} \right) \left( \frac{\sqrt{6\pi}}{2 \left( 3 - \sqrt{2} \right)^{3/2}} + \frac{9}{2 \left( 3 - \sqrt{2} \right)^2 \sqrt{n}} \right),
$$

where the last inequality follows from Lyapounov’s inequality, i.e.,

$$
\hat{\sigma} = E^{1/2} \left[ |X_{d+1} - \hat{\mu}|^2 \right] \leq E^{1/3} \left[ |X_{d+1} - \hat{\mu}|^3 \right] = \hat{\rho}^{1/3}.
$$
Taking (9) into (4), we finally obtain

\[
\pi \cdot \eta \leq \frac{1}{T} \left( \frac{\sqrt{2}n - 3}{n - 1} \right) \left( \frac{\sqrt{6\pi}}{2 (3 - \sqrt{2})^{3/2}} + \frac{9}{2} \frac{1}{(3 - \sqrt{2})^2 \sqrt{n}} \right) \\
+ \frac{12}{T \sqrt{2\pi}} h \left( \frac{T \sqrt{2\pi}}{2} \eta \right),
\]

or equivalently,

\[
\pi u - 6h(u) \leq \frac{\sqrt{\pi} (2n - 3\sqrt{2})}{4(n - 1)} \left( \frac{\sqrt{6\pi}}{(3 - \sqrt{2})^{3/2}} + \frac{9}{(3 - \sqrt{2})^2 \sqrt{n}} \right),
\]

(10)

for \( u \triangleq T \sqrt{2\pi} \eta / 2 \).
Observe that

\[ \pi u - 6h(u) = \pi u \left( 1 - 6 \int_u^\infty v_T \left( \frac{t}{T} \right) \frac{dt}{T} \right) \]

is continuous, and equals 0 at \( u = 0 \), and goes to \( \infty \) as \( u \to \infty \), which guarantees the existence of positive \( u \) satisfying (10).

Inequality (10) thus implies

\[
\begin{align*}
  u & \leq \max \left\{ a \geq 0 : \pi a - 6h(a) \leq \sqrt{\pi} \frac{2n - 3\sqrt{2}}{4(n - 1)} \left( \frac{\sqrt{6\pi}}{3 - \sqrt{2}} \right)^{3/2} + \frac{9}{(3 - \sqrt{2})^2 \sqrt{n}} \right\}.
\end{align*}
\]

Since

\[
\frac{\sqrt{\pi}(2n - 3\sqrt{2})}{4(n - 1)} = \frac{\sqrt{\pi}}{4} \left( 2 - \frac{3\sqrt{2} - 2}{n - 1} \right) \leq \frac{\sqrt{\pi}}{2},
\]

we can, for \( n \geq 3 \), further upper-bound \( u \) by:

\[
\begin{align*}
  u & \leq \max \left\{ a \geq 0 : \pi a - 6h(a) \leq \frac{\sqrt{\pi}}{2} \left( \frac{\sqrt{6\pi}}{3 - \sqrt{2}} \right)^{3/2} + \frac{9}{(3 - \sqrt{2})^2 \sqrt{3}} \right\} \\
  &= 3.26369 \ldots.
\end{align*}
\]
Berry-Esseen theorem and probability bound

The proof is completed by

\[ \eta = u \frac{2}{T \sqrt{2\pi}} \]
\[ \leq 3.26369 \frac{2}{T \sqrt{2\pi}} \]
\[ = 6.52738 \frac{(n-1)}{\sqrt{\pi (2n-3\sqrt{2}) \hat{\sigma}^3 \sqrt{n}}} \hat{\rho} \]
\[ = 6.52738 \frac{2}{2\sqrt{\pi}} \left( 1 + \frac{3\sqrt{2} - 2}{2n - 3\sqrt{2}} \right) \frac{\hat{\rho}}{\hat{\sigma}^3 \sqrt{n}} \]
\[ \leq 6.52738 \frac{2}{2\sqrt{\pi}} \left( 1 + \frac{3\sqrt{2} - 2}{6 - 3\sqrt{2}} \right) \frac{\hat{\rho}}{\hat{\sigma}^3 \sqrt{n}} \]
\[ = 4.19115 \frac{\hat{\rho}}{\hat{\sigma}^3 \sqrt{n}} \]
\[ \leq 5\frac{\hat{\rho}}{\hat{\sigma}^3 \sqrt{n}}. \]

□